

GLOBAL WEAK SOLUTIONS TO 3D COMPRESSIBLE NAVIER-STOKES-POISSON EQUATIONS WITH DENSITY-DEPENDENT VISCOSITY

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Abstract: Global-in-time weak solutions to the Compressible Navier-Stokes-Poisson equations in a three-dimensional torus for large data are considered in this paper. The system takes into account density-dependent viscosity and non-monotone pressure. We prove the existence of global weak solutions to NSP equations with damping term by using the Faedo-Galerkin method and the compactness arguments on the condition that the adiabatic constant satisfies $\gamma > \frac{4}{3}$.

Keywords: global weak solutions; compressible Navier-Stokes-Poisson equations; density-dependent viscosity; vacuum.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the following compressible Navier-Stokes-Poisson equations with density-dependent viscosity coefficients:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - \operatorname{div}(\rho \mathbb{D}u) + r_1 \rho |u| u = \rho \nabla \Phi, \\ \lambda \Delta \Phi = 4\pi G(\rho - \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} \rho dx), \quad \lambda = \pm 1, \end{cases} \quad (1.1)$$

with the initial conditions

$$\rho(0, x) = \rho_0, \rho u(0, x) = m_0, \quad (1.2)$$

where $t \geq 0, x \in \mathbb{T}^3$, $\rho = \rho(t, x)$ and $u = u(t, x)$ represent the fluid density and velocity respectively, $\mathbb{D}u$ is the strain tensor with $\mathbb{D}u = \frac{\nabla u + \nabla u^T}{2}$, and the pressure P is a non-monotone function of the density (see [7] for motivations) which satisfies the following conditions:

$$\begin{cases} P \in C^1(\mathbb{R}_+), \quad P(0) = 0, \\ \frac{1}{a} z^{\gamma-1} - b \leq P'(z) \leq a z^{\gamma-1} + b \quad \text{for all } z \geq 0, \end{cases} \quad (1.3)$$

for two constants $a > 0$ and $b \geq 0$.

The term on the right hand side of the second equation in (1.1) describes the internal force of gradient vector field produced by potential functions, which can be uniquely solved by the poisson equation (1.1)₃, yields

$$\lambda \Phi(x) = G \int g(x, y) \rho(y) dy,$$

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if and only if

$$\lambda \Delta \Phi = 4\pi G(\rho - \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} \rho dx), \quad \text{in } \mathbb{T}^3, \quad \int_{\mathbb{T}^3} \Phi dx = 0,$$

where $g = g(x, y)$ denotes the Green's function of the poisson part, $G > 0$ is a fixed constant. Moreover, for simplicity, using the conversation of mass $\int_{\mathbb{T}^3} \rho dx = \int_{\mathbb{T}^3} \rho_0 dx$, the poisson equation (1.1)₃ can be normalized as

$$\lambda \Delta \Phi = 4\pi G(\rho - 1).$$

From a physical point of view, the meaning of the Navier-Stokes-Poisson system is determined by the sign of the parameter λ . When $\lambda > 0$, the potential force Φ represents the electrostatic potential which produces the electric field $E = -\nabla \Phi$ and Equations (1.1) are used to describe the transportation of charged particles in electronic devices, and then $\rho \geq 0, u$ represent the charge density and velocity, respectively. On the other hand, if $\lambda < 0$, the potential force Φ denotes the gravitational force and the Navier-Stokes-Poisson system is used in astrophysics to describe the motion of gaseous stars, and $\rho \geq 0, u$ denote the density, velocity of a gaseous star, respectively.

Navier-Stokes-Poisson equations has attracted the attention of many physicists and mathematicians because of its physical importance, rich phenomena, and mathematical challenges. For the case of constant viscosity coefficients, Ducomet and Feireisl in [6] considered the full Navier-Stokes-Poisson equations and proved when $\gamma > \frac{3}{2}$, there exists a global-in-time variational weak solution. In [7] Ducomet et al. also proved there exists a global weak solution to the barotropic compressible Navier-Stokes-Poisson equations with no-monotone pressure provided that $\gamma > \frac{3}{2}$. Donatelli [5] considered the Cauchy problem for the coupled Navier-Stokes-Poisson equations and gave a positive answer to the existence of local and global weak solutions. Zhang and Tan in [24], by using the theory of Orlicz spaces, have proved the existence of globally defined finite energy weak solutions for Navier-Stokes-Poisson equations in two dimensions with the pressure satisfying $P(\rho) = a\rho \log^d \rho$ for large ρ , and $d > 1, a > 0$. Cai and Tan [4] also proved the system has the global weak time-periodic solution for the Navier-Stokes-Poisson equations in a bounded domain with periodic boundary condition as $\gamma > \frac{5}{3}$ when the external force is time-periodic. Besides, Jiang et al.[16] considered the global behavior of weak solutions of the Navie-Stokes-Poisson equations in a bounded domain with arbitrary forces.

However, for the case of density-dependent viscosity coefficients, the problem is much more challenge because of the degeneration near the vacuum and the results are limit. Ducomet et.al in [8] studied the global stability of the weak solutions to the Navier-Stokes-Poisson equations with the degenerate viscosities as $\gamma > \frac{4}{3}$, in which the pressure $P(\rho)$ is not necessary a monotone function of the density. Furthermore, in [9] they also considered Cauchy problem for the Navier-Stokes-Poisson equations of spherically symmetric motions in \mathbb{R}^3 , both constant viscosities and density-dependent viscosities included, and proved the global stability of the weak solutions just provided that the polytropic index γ satisfies $\gamma > 1$. Particularly, if without the drag term $\rho|u|u$ and the poisson term $\rho \nabla \Phi$, the equations (1.1)

will be reduced to the classical barotropic compressible Navier-Stokes equations with degenerate viscosities, especially to be the well known shallow water equation in the dimensions two, the global existence of weak solutions of which is a long standing open problem proposed by Lions [14]. Recently, A.Vasseur and C.Yu have proved in [21] that there exists a global weak solution to the compressible Navier-Stokes equations by constructing some smooth multipliers allowing to derive the Mellet-Vasseur type inequality [17] for the weak solutions, and they also proved the existence of the global weak solutions for the quantum compressible Navier-Stokes equations, which can be seen as a approximate system to the original equations in [21], almost at the same time, Li and Xin gave an another approach in [13]. Moreover, it should be noted that the Quasi-neutral limit problem is also considered by many mathematicians for the case of $\lambda > 0$, for more details, the readers can refer to [3, 23] and references therein.

Inspired by [17], in the present paper, we consider the compressible Navier-Stokes equations with degenerate viscosities, potential force and the damping term, in which the pressure $P(\rho)$ is not necessary a monotone function, and we prove that the problem admits a global weak solution as $\gamma > \frac{4}{3}$ for the case $\lambda = -1$, or $\gamma > 1$ for the case $\lambda = 1$. Compared to the case of the classical compressible Navier-Stokes equations, the Navier-Stokes-Poisson problem is much more complex and some new difficulties will occur, for example, we can not deduce the energy estimates directly due to the poisson term and non-monotone pressure term, and moreover, when we deduce the energy estimates and the Bresch-Dejardins entropy, the estimates will depend on the index ε, δ and η , so we need to be very careful as we deduce these estimates because we need to tend the $\varepsilon, \eta, \delta$ to zero step by step later in the proof of the main theorem. To our knowledge, this is the first complete proof for the Navier-Stokes-Poisson problem with the degenerate viscosities and here the pressure is not necessary a monotone function of the density, which contains the classical γ law case. So our results are much general and can be seen as a supplement of Ducomet et.al[8] and a extension of [21], [22].

Throughout this paper, we only focus on the case that $\lambda = -1$, and after some small modifications, the method can be directly applied to the case that $\lambda = 1$, so we omit the details.

1.1. Formulation of the weak solutions and main result. For the smooth solutions $(\rho, u, \Phi(\rho))$, multiplying the momentum equation $(1.1)_2$ and integrating by parts we can deduce the following energy inequality

$$E(t) + \int_0^T \int_{\mathbb{T}^3} \rho |\mathbb{D}u|^2 + r_1 \int_0^T \int_{\mathbb{T}^3} \rho u^3 \leq E_0, \quad (1.4)$$

where

$$E(t) = \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho u^2 + \Pi(\rho) - \frac{1}{8\pi G} |\nabla \Phi|^2 \right) dx, \quad \Pi(\rho) = \rho \int_1^\rho \frac{P(s)}{s^2} ds$$

and

$$E_0 = \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho_0 u_0^2 + \Pi(\rho_0) - \frac{1}{8\pi G} |\nabla \Phi(\rho_0)|^2 \right) dx.$$

However, the above energy estimate is not enough to prove the stability of the weak solutions $(\rho, u, \Phi(\rho))$ of (1.1), fortunately, if the viscosity coefficients satisfy a special relation, which means $\lambda(\rho) = \rho\mu(\rho)' - \mu(\rho)$, in this paper it means $\mu(\rho) = \rho$, $\lambda(\rho) = 0$, we will obtain the following B-D entropy estimate which was first introduced by Bresch-Desjardins-Lin in [2]:

$$\begin{aligned} & \int_{\mathbb{T}^3} \frac{1}{2} \rho \left(u + \frac{\nabla \rho}{\rho}\right)^2 dx + \frac{4}{a\gamma^2} \int_0^T \int_{\mathbb{T}^3} |\nabla \rho^{\frac{\gamma}{2}}|^2 dx dt \\ & + \int_0^T \int_{\mathbb{T}^3} \rho |\nabla u|^2 dx dt \leq \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho_0 u_0^2 + |\nabla \sqrt{\rho_0}|^2\right) dx + C, \end{aligned} \quad (1.5)$$

where C is bounded by the initial energy. Thus the initial data should satisfy the follows

$$\begin{aligned} & \rho_0 \in L^1(\mathbb{T}^3) \cap L^\gamma(\mathbb{T}^3), \quad \rho_0 \geq 0, \quad \nabla \sqrt{\rho_0} \in L^2(\mathbb{T}^3), \\ & m_0 \in L^1(\mathbb{T}^3), \quad m_0 = 0 \quad \text{if} \quad \rho_0 = 0, \quad \frac{|m_0|^2}{\rho_0} \in L^1(\mathbb{T}^3). \end{aligned} \quad (1.6)$$

Definition 1.1. We will say (ρ, u, Φ) is the finite energy weak solution of the problem (1.1)-(1.2) if the following is satisfied.

(1) ρ, u belong to the classes

$$\begin{cases} \rho \in L^\infty((0, T); L^1 \cap L^\gamma(\mathbb{T}^3)), \quad \sqrt{\rho} u \in L^\infty((0, T); L^2(\mathbb{T}^3)), \\ \sqrt{\rho} \in L^\infty((0, T); H^1(\mathbb{T}^3)), \quad \nabla \rho^{\frac{\gamma}{2}} \in L^2((0, T); L^2(\mathbb{T}^3)), \\ \sqrt{\rho} \nabla u \in L^2((0, T); L^2(\mathbb{T}^3)), \quad \rho^{\frac{1}{3}} u \in L^3((0, T); L^3(\mathbb{T}^3)), \end{cases} \quad (1.7)$$

- (2) The equations (1.1)₁-(1.1)₂ hold in the sense of $\mathcal{D}'((0, T) \times \mathbb{T}^3)$, (1.1)₃ holds a.e. for $(t, x) \in ((0, T) \times \mathbb{T}^3)$,
(3) (1.2) holds in $\mathcal{D}'(\mathbb{T}^3)$,
(4) (1.4) and (1.5) hold for almost every $t \in [0, T]$,

Before stating the main theorem, we will give some notations:

Notations: Throughout this paper, C denotes a generic positive constant which may depend on the initial data or some other constants but independent of the indexes $\varepsilon, \eta, \delta$ and r_0 , $C(\cdot) > 0$ means the constant C particularly depends on the parameters in the bracket, and

$$\int f = \int_{\mathbb{T}^3} f dx, \quad \int_0^T \int f = \int_0^T \int_{\mathbb{T}^3} f dx dt,$$

$$\|f\|_{L^p} = \|f\|_{L^p(\mathbb{T}^3)}, \quad \|f\|_{L^p(0, T; W^{s, r})} = \|f\|_{L^p(0, T; W^{s, r}(\mathbb{T}^3))}.$$

Then we will state our main results:

Theorem 1.1. Let $\lambda = -1$, $\gamma > \frac{4}{3}$ and the initial data satisfies (1.6), then for any time T , there exists a weak solution (ρ, u, Φ) to (1.1)-(1.2) in the sense of Definition 1.1.

Remark 1.1. *It should be pointed out that the damping term $\rho|u|u$ here is used to give the strong convergence of $\sqrt{\rho}u$ in $L^2(0, T; L^2(\mathbb{T}^3))$, so follow the approach in [21], we can similarly deduce the Mellet-Vasseur inequality for weak solutions, and the damping term can be removed, for this case, we will show the details in our future paper.*

Remark 1.2. *For the case that $\lambda = 1$, with some small modifications, the proof in this paper can be directly extended to this case, and we can also prove that the system (1.1) admits a global weak solution just provided that $\gamma > 1$. So in this paper, we omit the details.*

The rest paper is organized as follows: in section 2, we state some elementary inequalities and compactness theorems which will be used frequently in the whole proof. To proof our main result, we use the weak compactness analysis method and need to pass to the limits at several approximate levels. In section 3, following the method in [22], we show the existence of global-in-time weak solutions to the approximate equations by using the Faedo-Galerkin method. In section 4, we deduce the Bresch-Dejardins entropy estimates and pass to the limits as $\varepsilon, \mu \rightarrow 0$. In section 5-6, by using the standard compactness arguments, we pass to the limits as $\eta \rightarrow 0$ and $\delta \rightarrow 0$ step by step.

2. PRELIMINARIES

Firstly, we introduce the following Gagliardo-Nirenberg inequality which we will used later when we deduce the energy estimates and B-D entropy.

Lemma 2.1. [18] (*Gagliardo-Nirenberg interpolation inequality*) *For function $u : \Omega \rightarrow \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $\forall 1 \leq q, r \leq \infty$ and a natural number m , Suppose also that a real number α and a natural number j are such that*

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1-\alpha}{q}$$

and

$$\frac{j}{m} \leq \alpha \leq 1,$$

then we have

$$\|D^j u\|_{L^p} \leq C_1 \|D^m u\|_{L^r}^\alpha \|u\|_{L^q}^{1-\alpha} + C_2 \|u\|_{L^s}$$

where $s > 0$ is arbitrary; naturally, the constants C_1 and C_2 depend upon the domain Ω as well as m, n etc.

The following two Lemmas are two standard compactness results and will help us get the strong convergence of the solutions:

Lemma 2.2. [1, 20] (*Aubin-Lions Lemma*) *Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1 . For $1 \leq p, q \leq +\infty$, let*

$$W = \{u \in L^p([0, T]; X_0) \mid \partial_t u \in L^q([0, T]; X_1)\}.$$

- (i) If $p < +\infty$, then the embedding of W into $L^p([0, T]; X)$ is compact.
(ii) If $p = +\infty$ and $q > 1$, then the embedding of W into $C([0, T]; X)$ is compact

Lemma 2.3. [19] (Egoroff's theorem about uniform convergence) Let $f_n \rightarrow f$ a.e. in Ω , a bounded measurable set in \mathbb{R}^n , with f finite a.e. Then for any $\varepsilon > 0$ there exists a measurable subset $\Omega_\varepsilon \subset \Omega$ such that $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ and $f_n \rightarrow f$ uniformly in Ω_ε , moreover, if

$$f_n \rightarrow f \text{ a.e. in } \Omega, \\ f_n \in L^p(\Omega) \text{ and uniformly bounded, for any } 1 < p \leq +\infty,$$

then, we have

$$f_n \rightarrow f \text{ strongly in } L^s, \text{ for any } s \in [1, p).$$

Proof. Since $f_n \rightarrow f$ a.e. in Ω and f_n is uniformly bounded in $L^p(\Omega)$, so due to the Egoroff's theorem, we have

$$\forall \varepsilon > 0, \exists \Omega_\varepsilon, |\Omega - \Omega_\varepsilon| < \varepsilon, \sup_{x \in \Omega_\varepsilon} |f_n(x) - f(x)| \rightarrow 0, \text{ uniformly with } n,$$

then we can get

$$\begin{aligned} \int_{\Omega} |f_n - f|^s dx &= \int_{\Omega_\varepsilon} |f_n - f|^s dx + \int_{\Omega - \Omega_\varepsilon} |f_n - f|^s dx \\ &\leq \sup_{x \in \Omega_\varepsilon} |f_n - f|^s |\Omega_\varepsilon| + C \|f_n - f\|_{L^p}^s |\Omega - \Omega_\varepsilon|^{(p-s)/p} \rightarrow 0. \end{aligned}$$

□

3. FAEDO-GALERKIN APPROXIMATION

In this section, we construct the approximate system to the original problem (1.1) by using the Faedo-Galerkin method, we proceed similarly in [[10], Chapter. 7] and [15].

3.1. Approximate the mass equation. Let $T > 0$, then we define a finite-dimensional space $X_n = \text{span}\{e_1, \dots, e_n\}$, $n \in \mathbb{N}$, where $\{e_k\}$ is an orthonormal basis of $L^2(\mathbb{T}^3)$ which is also an orthogonal basis of $H^1(\mathbb{T}^3)$. Let $(\rho_0, u_0) \in C^\infty(\mathbb{T}^3)$ be some initial data satisfying $\rho_0 \geq \nu > 0$ for $x \in (\mathbb{T}^3)$ for some $\nu > 0$, and let the velocity $u \in C([0, T]; X_n)$ be given with the following norm

$$u(x, t) = \sum_{i=1}^n \lambda_i(t) e_i(x), \quad (t, x) \in [0, T] \times \mathbb{T}^3$$

Note that X_n is a finite-dimensional space, all the norms are equivalence on X_n , so u is bounded in $C([0, T]; C^k(\mathbb{T}^3))$ for any $k \in \mathbb{N}$ and there exists a constant $C > 0$ depending on k such that

$$\|u\|_{C([0, T]; C^k(\mathbb{T}^3))} \leq C \|u\|_{C([0, T]; L^2(\mathbb{T}^3))}. \quad (3.1)$$

Then we approximate the continuity equation as follows:

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = \varepsilon \Delta \rho, \\ \rho_0 \in C^\infty(\mathbb{T}^3), \quad \rho_0 \geq \nu > 0, \end{cases} \quad (3.2)$$

Firstly, to show the well-posedness of the parabolic problem (3.2), we introduce the following Lemma:

Lemma 3.1. [11] *Let $\Omega \in \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0, 1)$ and let $u \in C([0, T]; X_n)$ be a given vector field. If the initial data $\rho_0 \geq \nu > 0$, $\rho_0 \in C^2(\mathbb{T}^3)$, then problem (3.2) possesses a unique classical solution $\rho = \rho_u$, more specifically,*

$$\rho_u \in V \equiv \left\{ \rho \in C([0, T]; C^{2,\nu}(\mathbb{T}^3)), \right. \\ \left. \partial_t \rho \in C([0, T]; C^{0,\nu}(\mathbb{T}^3)). \right\} \quad (3.3)$$

Furthermore, because $u \in C([0, T]; X_n)$ is a given vector field, then by using the bootstrap method and above Lemma, it's easy to prove that the system (3.2) exists an unique classical solution $\rho \in C^1([0, T]; C^7(\mathbb{T}^3))$. Moreover, if $0 < \underline{\rho} \leq \rho \leq \bar{\rho}$ and $\text{div} u \in L^1([0, T]; L^\infty(\mathbb{T}^3))$, through the maximum principle it provides

$$\rho(x, t) \geq 0.$$

Then if we define $L\rho = \partial_t \rho + \text{div}(\rho u) - \varepsilon \Delta \rho$, by direct calculation we can obtain

$$L(\bar{\rho} e^{\int_0^T \|\text{div} u\|_{L^\infty} dt}) = \bar{\rho} e^{\int_0^T \|\text{div} u\|_{L^\infty} dt} (\|\text{div} u\|_{L^\infty} + \text{div} u) \geq 0,$$

$$L(\underline{\rho} e^{-\int_0^T \|\text{div} u\|_{L^\infty} dt}) \leq 0, \quad L\rho = 0,$$

which means that $\bar{\rho} e^{\int_0^T \|\text{div} u\|_{L^\infty} dt}$ and $\underline{\rho} e^{-\int_0^T \|\text{div} u\|_{L^\infty} dt}$ are super and sub solutions to the equation (3.2) respectively, for $0 < \underline{\rho} \leq \rho \leq \bar{\rho}$. So, by using the comparison principle, we can obtain

$$0 < \underline{\rho} e^{-\int_0^T \|\text{div} u\|_{L^\infty} dt} \leq \rho(x, t) \leq \bar{\rho} e^{\int_0^T \|\text{div} u\|_{L^\infty} dt}, \quad \forall x \in \mathbb{T}^3, \quad t \geq 0. \quad (3.4)$$

Next we will show that the solution of the equation (3.2) depends on the velocity u continuously. Let ρ_1, ρ_2 are two solutions with the same initial data, which means

$$\partial_t \rho + \text{div}(\rho_1 u_1) = \varepsilon \Delta \rho_1, \quad \partial_t \rho + \text{div}(\rho_2 u_2) = \varepsilon \Delta \rho_2.$$

Subtracting the above two equations, multiplying the result equation with $-\Delta(\rho_1 - \rho_2)$ and integrating by parts with respect to x over \mathbb{T}^3 , we have

$$\sup_{t \in [0, \tau]} \|\rho_1 - \rho_2\|_{H^1} \leq \tau C(\rho_0, \varepsilon, \|u_1\|_{L^1((0, \tau); W^{1, \infty})}, \|u_1\|_{L^1((0, \tau); W^{1, \infty})}) \|u_1 - u_2\|_{H^1},$$

moreover, for $u \in C([0, T]; X_n)$ is a given vector field, similarly by using the bootstrap method and compactness analysis, we can prove

$$\|\rho_1 - \rho_2\|_{C([0, \tau]; C^7(\mathbb{T}^3))} \\ \leq \tau C(\rho_0, \varepsilon, \|u_1\|_{L^1((0, \tau); X_n)}, \|u_1\|_{L^1((0, \tau); X_n)}) \|u_1 - u_2\|_{C([0, \tau]; X_n)}. \quad (3.5)$$

So if we introduce the operator $\mathcal{S} : C([0, T]; X_n) \rightarrow C([0, T]; C^7(\mathbb{T}^3))$ by $\mathcal{S}(u) = \rho$, we have the following Proposition

Proposition 3.1. *If $0 < \underline{\rho} \leq \rho \leq \bar{\rho}$, $\rho_0 \in C^\infty(\mathbb{T}^3)$, $u \in C([0, T]; X_n)$, then there exists an operator $\mathcal{S} : C([0, T]; X_n) \rightarrow C([0, T]; C^7(\mathbb{T}^3))$ satisfying*

- $\rho = \mathcal{S}(\rho)$ is an unique solution to the problem (3.2).
- $0 < \underline{\rho} e^{-\int_0^T \|\text{div} u\|_{L^\infty} dt} \leq \rho(x, t) \leq \bar{\rho} e^{\int_0^T \|\text{div} u\|_{L^\infty} dt}, \quad \forall x \in \mathbb{T}^3, \quad t \geq 0.$

- $\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{C([0,\tau];C^7(\mathbb{T}^3))}$
 $\leq \tau C(\rho_0, \varepsilon, \|u_1\|_{L^1((0,\tau);X_n)}, \|u_1\|_{L^1((0,T\tau);X_n)}) \|u_1 - u_2\|_{C([0,\tau];X_n)},$ for any $\tau \in [0, T]$ and $u_1, u_2 \in M_k = \{u \in C([0, T]; X_n); \|u\|_{C([0,T];X_n)} \leq k, t \in [0, T]\}$.

Remark 3.1. The proposition 3.1 shows the operator \mathcal{S} is also Lipschitz continuous for sufficient small time t .

3.2. Faedo-Galerkin approximation. Next we wish to solve the momentum equation on the space X_n by using the Faedo-Galerkin approximation method. To this end, for given $\rho = \mathcal{S}(u)$, we are looking for a approximate solution $u_n \in C([0, T]; X_n)$ satisfying

$$\begin{aligned} & \int_{\mathbb{T}^3} \rho u_n(T) \varphi dx - \int_{\mathbb{T}^3} m_0 \varphi dx + \mu \int_0^T \int_{\mathbb{T}^3} \Delta u_n \cdot \Delta \varphi dx dt \\ & - \int_0^T \int_{\mathbb{T}^3} (\rho u_n \otimes u_n) \cdot \nabla \varphi dx dt - \int_0^T \int_{\mathbb{T}^3} \rho \mathbb{D} u_n \cdot \nabla \varphi dx dt - \int_0^T \int_{\mathbb{T}^3} P(\rho) \operatorname{div} \varphi dx dt \\ & + \eta \int_0^T \int_{\mathbb{T}^3} \rho^{-6} \operatorname{div} \varphi dx dt + \varepsilon \int_0^T \int_{\mathbb{T}^3} \nabla \rho \cdot \nabla u_n \varphi dx dt + r_0 \int_0^T \int_{\mathbb{T}^3} u_n \varphi dx dt \\ & + r_1 \int_0^T \int_{\mathbb{T}^3} \rho |u_n| u_n \varphi dx dt - \delta \int_0^T \int_{\mathbb{T}^3} \rho \nabla \Delta^3 \rho \varphi dx dt = \int_0^T \int_{\mathbb{T}^3} \rho \nabla \Phi \varphi dx dt, \end{aligned} \quad (3.6)$$

for any test function $\varphi \in X_n$. The extra term $\mu \Delta^2 u_n$ is not only necessary to extend the local solution obtained by the fixed point theorem to a global one at the Galerkin level but also to make sure $\partial_t(\frac{\nabla \rho}{\rho}) \in L^2((0, T); L^2)$ so that it can be taken as a test function when we compute the B-D entropy at the next level, and the extra terms $\eta \nabla \rho^{-6}$ and $\delta \rho \nabla \Delta^3 \rho$ are also necessary to keep the density bounded, and bounded away from below with a positive constant for all the time, this enables us to take $\frac{\nabla \rho}{\rho}$ as a test function to derive the B-D entropy, and the term $r_0 u_n$ is used to control the density near the vacuum, $\rho |u_n| u_n$ is used to make sure that $\sqrt{\rho} u$ is strong convergence in $L^2(0, T; L^2(\mathbb{T}^3))$ at the last approximation level.

To solve (3.6), we follow the same arguments as in [10, 12, 15], and introduce the following operators, giving a function $\rho \in L^1(\mathbb{T}^3)$ with $\rho > \underline{\rho} > 0$:

$$\mathcal{M}[\rho] : X_n \rightarrow X_n^*, \quad \langle \mathcal{M}[\rho] u, v \rangle = \int_{\mathbb{T}^3} \rho u \cdot v dx, \quad u, v \in X_n.$$

Similarly in [12], it's easy to check that the operator $\mathcal{M}[\rho]$ satisfies the following properties:

- $\|\mathcal{M}[\rho]\|_{\mathcal{L}(X_n, X_n^*)} \leq C(n) \|\rho\|_{L^1}.$
- $\|\mathcal{M}[\rho]\|_{\mathcal{L}(X_n, X_n^*)} \geq \inf_{x \in \mathbb{T}^3} \rho$
- If $\inf_{x \in \mathbb{T}^3} \rho \geq \underline{\rho} > 0$, then the operator is invertible with

$$\|\mathcal{M}^{-1}[\rho]\|_{\mathcal{L}(X_n^*, X_n)} \leq \underline{\rho}^{-1},$$

where $\mathcal{L}(X_n^*, X_n)$ is the set of bounded linear mappings from X_n^* to X_n .

- $\mathcal{M}^{-1}[\rho]$ is Lipschitz continuous in the sense

$$\|\mathcal{M}^{-1}[\rho_1] - \mathcal{M}^{-1}[\rho_2]\|_{\mathcal{L}(X_n^*, X_n)} \leq C(n, \underline{\rho}) \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^3)}$$

for all $\rho_1, \rho_2 \in L^1(\mathbb{T}^3)$ such that $\rho_1, \rho_2 \geq \underline{\rho} > 0$.

Proof. Here, we omit the proof, for more details, we refer the readers to [10, 12, 15]. \square

Then by using the operators $\mathcal{M}[\rho]$ and $\rho = \mathcal{S}(u_n)$, we rewrite (3.6) as the following fixed-point problem

$$u_n(t) = \mathcal{M}^{-1}[(\mathcal{S}(u_n)(t))(\mathcal{M}[\rho_0](u_0) + \int_0^T \mathcal{N}(\mathcal{S}(u_n), u_n)(s)ds), \quad (3.7)$$

where

$$\begin{aligned} \mathcal{N}(\mathcal{S}(u_n), u_n)(s) = & \rho \nabla \Phi - \operatorname{div}(\rho u_n \otimes u_n) + \operatorname{div}(\rho \mathbb{D} u_n) - \mu \Delta^2 u_n - \varepsilon \nabla \rho \cdot \nabla u_n \\ & - \nabla P(\rho) + \eta \nabla \rho^{-6} - r_0 u_n - r_1 \rho |u_n| u_n + \delta \rho \nabla \Delta^3 \rho. \end{aligned}$$

Thanks to the Lipschitz continuous estimates for \mathcal{S} and \mathcal{M}^{-1} , this equation can be solved by using the fixed-point theorem of Banach for a short time $[0, T']$, where $T' \leq T$, in the space $C([0, T]; X_n)$. Thus there exists a unique local-in-time solution $(\rho_n, u_n, \Phi(\rho_n))$ to (3.2) and (3.6). Next we will extend this obtained local solution to be a global one.

Differentiating (3.6) with respect to time t , taking $\phi = u_n$ and integrating by parts with respect to x over \mathbb{T}^3 , we have the following energy estimate

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho u_n^2 + \frac{1}{2} \int \rho_t |u_n|^2 + \int \operatorname{div}(\rho u_n) |u_n|^2 + \int \rho u_n \cdot \nabla u_n : u_n \\ & + \int \nabla P(\rho) \cdot u_n - \eta \int \nabla \rho^{-6} \cdot u_n + \varepsilon \int \nabla \rho \cdot \nabla u_n \cdot u_n + \int \rho |\mathbb{D} u_n|^2 + r_0 \int |u_n|^2 \\ & + r_1 \int \rho |u_n|^3 + \delta \int \operatorname{div}(\rho u_n) \Delta^3 \rho + \mu \int |\Delta u_n|^2 = - \int \operatorname{div}(\rho u_n) \Phi, \end{aligned} \quad (3.8)$$

firstly, we estimate the terms on the left hand side one by one:

$$\begin{aligned} & \frac{1}{2} \int \rho_t |u_n|^2 + \int \operatorname{div}(\rho u) |u_n|^2 + \int \rho u_n \cdot \nabla u_n : u_n \\ & = \frac{1}{2} \int [\varepsilon \Delta \rho - \operatorname{div}(\rho u_n)] |u_n|^2 + \int \operatorname{div}(\rho u) |u_n|^2 + \int \rho u_n^i \partial_i u_n^j u_n^j \\ & = \frac{1}{2} \int \operatorname{div}(\rho u_n) |u_n|^2 - \varepsilon \int \nabla \rho \cdot \nabla u_n \cdot u_n - \frac{1}{2} \int \operatorname{div}(\rho u_n) |u_n|^2 \\ & = -\varepsilon \int \nabla \rho \cdot \nabla u_n \cdot u_n, \end{aligned} \quad (3.9)$$

where we used the approximate mass equation (3.2).

$$\begin{aligned} \int \nabla P(\rho) \cdot u_n & = \int \nabla \int_1^\rho \frac{P'(s)}{s} ds (\rho u_n) dx = - \int \int_1^\rho \frac{P'(s)}{s} ds [\varepsilon \Delta \rho - \rho_t] dx \\ & = \frac{d}{dt} \int \Pi(\rho) dx + \varepsilon \int \frac{P'(\rho)}{\rho} |\nabla \rho|^2, \end{aligned} \quad (3.10)$$

where we used (3.2) and integration by parts, and $\Pi(\rho) = \rho \int_1^\rho \frac{P(s)}{s^2} ds$. Next we will deal with the cold pressure and high order derivative of the density terms as follows

$$\begin{aligned} -\eta \int \nabla \rho^{-6} \cdot u_n &= -\frac{6}{7}\eta \int \rho u_n \cdot \nabla \rho^{-7} = \frac{6}{7}\eta \int \rho^{-7} [\varepsilon \Delta \rho - \rho_t] \\ &= \frac{1}{7}\eta \frac{d}{dt} \int \rho^{-6} + \frac{2}{3}\eta \varepsilon \int |\nabla \rho^{-3}|^2, \end{aligned} \quad (3.11)$$

$$\begin{aligned} \delta \int \operatorname{div}(\rho u_n) \Delta^3 \rho &= \delta \int [\varepsilon \Delta \rho - \rho_t] \Delta^3 \rho \\ &= \frac{\delta}{2} \frac{d}{dt} \int |\nabla \Delta \rho|^2 + \delta \varepsilon \int |\Delta^2 \rho|^2, \end{aligned} \quad (3.12)$$

finally, we will estimate the poisson term on the right hand side

$$\begin{aligned} - \int \operatorname{div}(\rho u_n) \Phi &= - \int [\varepsilon \Delta \rho - \rho_t] \Phi = \int \partial_t(\rho - 1) \Phi - \varepsilon \int \Delta(\rho - 1) \Phi \\ &= - \int \frac{1}{4\pi G} \partial_t \Delta \Phi \cdot \Phi - \varepsilon \int (\rho - 1) \Delta \Phi \\ &= \frac{1}{8\pi G} \frac{d}{dt} \int |\nabla \Phi|^2 + \varepsilon \int 4\pi G (\rho - 1)^2, \end{aligned} \quad (3.13)$$

where we used the equation (1.1)₃.

Then substituting (3.9)-(3.13) into (3.8) and integrating the result equation with respect to t over $[0, T]$, yields

$$\begin{aligned} E(t) + \varepsilon \int_0^T \int \frac{P'(\rho)}{\rho} |\nabla \rho|^2 + \frac{2}{3}\eta \varepsilon \int_0^T \int |\nabla \rho^{-3}|^2 + \int_0^T \int \rho |\mathbb{D} u_n|^2 \\ + r_0 \int_0^T \int |u_n|^2 + r_1 \int_0^T \int \rho |u_n|^3 + \mu \int_0^T \int |\Delta u_n|^2 + \delta \varepsilon \int_0^T \int |\Delta^2 \rho|^2 \\ = 4\pi G \varepsilon \int_0^T \int (\rho - 1)^2 + E_0, \end{aligned} \quad (3.14)$$

where

$$E(t) = \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho u_n^2 + \Pi(\rho) + \frac{\eta}{7} \rho^{-6} - \frac{1}{8\pi G} |\nabla \Phi(\rho)|^2 + \frac{\delta}{2} |\nabla \Delta \rho|^2 \right) dx, \quad \Pi(\rho) = \rho \int_1^\rho \frac{P(s)}{s^2} ds$$

and

$$E_0 = \int_{\mathbb{T}^3} \left(\frac{1}{2} \rho_0 u_n^2 + \Pi(\rho_0) + \frac{\eta}{7} \rho_0^{-6} - \frac{1}{8\pi G} |\nabla \Phi(\rho_0)|^2 + \frac{\delta}{2} |\nabla \Delta \rho_0|^2 \right) dx.$$

Moreover, because of (1.3), we have

$$\Pi(\rho) = \rho \int_1^\rho \frac{P(s)}{s^2} ds \geq \rho \int_1^\rho \frac{\frac{1}{a\gamma} s^\gamma - bs}{s^2} ds = \frac{1}{a\gamma(\gamma-1)} [\rho^\gamma - \rho] - b\rho \log \rho, \quad (3.15)$$

and

$$\begin{aligned} \varepsilon \int_0^T \int \frac{P'(\rho)}{\rho} |\nabla \rho|^2 &\geq \varepsilon \int_0^T \int \frac{\frac{1}{a}\rho^{\gamma-1} - b}{\rho} |\nabla \rho|^2 \\ &= \frac{4\varepsilon}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 - b\varepsilon \int_0^T \int \frac{1}{\rho} |\nabla \rho|^2, \end{aligned} \quad (3.16)$$

furthermore, if $\gamma > \frac{4}{3}$, we also have

$$\begin{aligned} \frac{1}{8\pi G} \int |\nabla \Phi|^2 &\leq C \|\nabla \Phi\|_{L^2}^2 \leq C \|\nabla^2 \Phi\|_{L^{\frac{6}{5}}}^2 \leq C \|\rho - 1\|_{L^{\frac{6}{5}}}^2 \\ &\leq C(1 + \|\rho\|_{L^{\frac{6}{5}}}^2) \leq C(1 + \|\rho\|_{L^1}^{\frac{5\gamma-6}{3(\gamma-1)}} \|\rho\|_{L^\gamma}^{\frac{\gamma}{3(\gamma-1)}}) \leq C + \varsigma \|\rho\|_{L^\gamma}^\gamma, \end{aligned} \quad (3.17)$$

where $0 < \varsigma \ll 1$ is a fixed constant, C is a generic positive constant independent of $\varepsilon, \eta, \delta, r_0$, and we also used conservation of mass, Sobolev inequality, Young inequality.

Then substituting (3.15)-(3.17) into (3.14), we have

$$\begin{aligned} &\int \frac{1}{2} \rho u_n^2 + \frac{1}{a\gamma(\gamma-1)} \rho^\gamma + \frac{\eta}{7} \rho^{-6} + \frac{\delta}{2} |\nabla \Delta \rho|^2 dx + \frac{4\varepsilon}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 \\ &+ \frac{2}{3} \eta \varepsilon \int_0^T \int |\nabla \rho^{-3}|^2 + \int_0^T \int \rho |\mathbb{D} u_n|^2 + r_0 \int_0^T \int |u_n|^2 + r_1 \int_0^T \int \rho |u_n|^3 \\ &+ \mu \int_0^T \int |\Delta u_n|^2 + \delta \varepsilon \int_0^T \int |\Delta^2 \rho|^2 \\ &= 4\pi G \varepsilon \int_0^T \int (\rho - 1)^2 + \int \frac{1}{a\gamma(\gamma-1)} \rho + b \int \rho \log \rho + C + \varsigma \|\rho\|_{L^\gamma}^\gamma \\ &+ b\varepsilon \int \frac{1}{\rho} |\nabla \rho|^2 \leq \varsigma' \|\rho\|_{L^\gamma}^\gamma + C + C\varepsilon \int_0^T \int \rho^2 + b\varepsilon \int \frac{1}{\rho} |\nabla \rho|^2, \end{aligned} \quad (3.18)$$

where ς' is a sufficient small positive constant, C is a generic positive constant only depending on the initial data and T .

Because

$$C\varepsilon \int_0^T \int \rho^2 \leq C\varepsilon \int_0^T \|\rho\|_{L^1}^{\frac{4}{3}} \|\nabla^3 \rho\|_{L^2}^{\frac{2}{3}} dt \leq C + C\frac{\varepsilon}{\delta} \int_0^T \delta \|\nabla^3 \rho\|_{L^2}^2 dt \quad (3.19)$$

and

$$\begin{aligned} b\varepsilon \int_0^T \int \frac{1}{\rho} |\nabla \rho|^2 &\leq C\varepsilon \int_0^T \|\rho^{-1}\|_{L^6} \|(\nabla \rho)^2\|_{L^{\frac{6}{5}}} dt \\ &\leq C\varepsilon \int_0^T \|\rho^{-1}\|_{L^6} \|\rho\|_{L^1}^{\frac{11}{9}} \|\nabla^3 \rho\|_{L^2}^{\frac{7}{9}} dt \\ &\leq C + \frac{C\varepsilon}{\eta} \int_0^T \eta \|\rho^{-1}\|_{L^6}^6 dt + \frac{C\varepsilon}{\delta} \int_0^T \delta \|\nabla^3 \rho\|_{L^2}^2 dt, \end{aligned} \quad (3.20)$$

then substituting (3.19)-(3.20) into (3.18) and using the Gronwall inequality gives

$$\int \eta \rho^{-6} + \delta |\nabla \Delta \rho|^2 dx \leq C + C\left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta}\right) T e^{C\left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta}\right)T}. \quad (3.21)$$

In combination with (3.18) and (3.21), we have the following energy inequality

$$\begin{aligned}
& \int \frac{1}{2} \rho u_n^2 + \frac{1}{a\gamma(\gamma-1)} \rho^\gamma + \frac{\eta}{7} \rho^{-6} + \frac{\delta}{2} |\nabla \Delta \rho|^2 dx + \frac{4\varepsilon}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 \\
& + \frac{2}{3} \eta \varepsilon \int_0^T \int |\nabla \rho^{-3}|^2 + \int_0^T \int \rho |\mathbb{D} u_n|^2 + r_0 \int_0^T \int |u_n|^2 + r_1 \int_0^T \int \rho |u_n|^3 \\
& + \mu \int_0^T \int |\Delta u_n|^2 + \delta \varepsilon \int_0^T \int |\Delta^2 \rho|^2 \\
& \leq C + C\left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta}\right) \left[1 + \left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta}\right) T e^{C\left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta}\right) T}\right] T
\end{aligned} \tag{3.22}$$

So the energy inequality (3.22) yields

$$\int_0^T \|\Delta u_n\|_{L^2}^2 dt \leq C(\varepsilon, \eta, \delta) < +\infty,$$

where $C(\varepsilon, \eta, \delta)$ denotes a positive constant especially depending on $\varepsilon, \eta, \delta$ but independent of n , and due to $\dim X_n < \infty$ and (3.4), then the density is bounded and bounded away from blow with a positive constant, which means there exists a constant $c > 0$ such that

$$0 < \frac{1}{c} \leq \rho_n \leq c, \tag{3.23}$$

for all $t \in [0, T^*)$. Furthermore, the energy inequality also gives us

$$\sup_{t \in (0, T^*)} \int \rho_n u_n^2 \leq C < \infty, \tag{3.24}$$

which together with (3.23) and (3.24), implies

$$\sup_{t \in (0, T^*)} \int \|u_n\|_{L^\infty} \leq C < \infty,$$

where we used the fact that all the norms are equivalence on X_n . Then we can repeat above argument many times and use the compactness analysis, we can obtain $u_n \in C([0, T]; X_n)$, so we can extend T^* to T . Thus there exists a global solution $(\rho_n, u_n, \Phi(\rho_n))$ to (3.2), (3.6) for any time T .

To conclude this part, we have the following proposition on the approximate solutions $(\rho_n, u_n, \Phi(\rho_n))$:

Proposition 3.2. *Let $(\rho_n, u_n, \Phi(\rho_n))$ be the solutions of (3.2), (3.6) on $(0, T) \times \mathbb{T}^3$ constructed above, then the solutions must satisfy the energy inequality (3.22). In particular, we have the following estimates*

$$\begin{aligned}
& \sqrt{\rho_n} u_n \in L^\infty(0, T; L^2), \sqrt{\rho} \mathbb{D} u_n \in L^2(0, T; L^2), \sqrt{\mu} \Delta u_n \in L^2(0, T; L^2), \\
& \rho_n \in L^\infty(0, T; L^1 \cap L^\gamma), \sqrt{\varepsilon} \nabla \rho_n^{\frac{\gamma}{2}} \in L^2(0, T; L^2), \eta \rho_n^{-6} \in L^\infty(0, T; L^1), \\
& \sqrt{\varepsilon \eta} \nabla \rho_n^{-3} \in L^2(0, T; L^2), \sqrt{\delta} \rho_n \in L^\infty(0, T; H^3), \sqrt{\delta \varepsilon} \rho_n \in L^2(0, T; H^4), \\
& \sqrt{r_0} u_n \in L^2(0, T; L^2), \rho_n^{\frac{1}{3}} u_n \in L^3(0, T; L^3).
\end{aligned} \tag{3.25}$$

3.3. Passing to the limits as $n \rightarrow \infty$. We perform first the limit as $n \rightarrow \infty$, $\varepsilon, \eta, \delta, r_0 > 0$ being fixed. Based on the above estimates which are uniform on n and Aubin-Lions Lemma, we have the following compactness results.

3.3.1. Step 1. Convergence of ρ_n , Pressure $P(\rho_n)$ and gravitational force $\nabla\Phi(\rho_n)$.

Lemma 3.2. *The following estimates hold for any fixed positive constants $\varepsilon, \eta, \delta$ and r_0 :*

$$\begin{aligned} \|\rho_n\|_{L^\infty(0,T;H^3)} + \|\rho_n\|_{L^2(0,T;H^4)} &\leq K, \quad \|\partial_t \rho_n\|_{L^2(0,T;H^{-1})} \leq K, \\ \|\rho_n^\gamma\|_{L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}})} &\leq K, \quad \|\rho_n^{-6}\|_{L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}})} \leq K, \\ \|\nabla\Phi(\rho_n)\|_{C([0,T];L^2)} &\leq C\|\rho - 1\|_{C([0,T];L^{\frac{6}{5}})} \leq K \end{aligned} \quad (3.26)$$

where K is independent of n , depends on $\varepsilon, \eta, \delta, r_0$, initial data and T . Moreover, up to an extracted subsequence

$$\begin{aligned} \rho_n &\rightarrow \rho \text{ a.e. and strongly in } C([0,T];H^3), \\ P(\rho_n) &\rightarrow P(\rho) \text{ a.e. and strongly in } L^1(0,T;L^1), \\ \rho_n^{-6} &\rightarrow \rho^{-6} \text{ a.e. and strongly in } L^1(0,T;L^1), \\ \nabla\Phi(\rho_n) &\rightarrow \nabla\Phi(\rho) \text{ strongly in } L^2(0,T;L^2). \end{aligned} \quad (3.27)$$

Proof. By (3.2), we have

$$\begin{aligned} \int_0^T \int (\rho_n)_t \varphi &= -\varepsilon \int_0^T \int \nabla \rho_n \nabla \varphi + \int_0^T \int (\rho_n u_n) \nabla \varphi \\ &\leq C \|\nabla \rho_n\|_{L^2(0,T;L^2)} \|\nabla \varphi\|_{L^2(0,T;L^2)} \\ &\quad + C \|\rho_n\|_{L^\infty(0,T;L^\infty)} \|u_n\|_{L^2(0,T;L^2)} \|\nabla \varphi\|_{L^2(0,T;L^2)} \leq C, \end{aligned}$$

holds for any $\varphi \in L^2(0,T;H^1)$, which yields $\partial_t \rho_n \in L^2(0,T;H^{-1})$. This together with $\rho_n \in L^\infty(0,T;H^3) \cap L^2(0,T;H^4)$, using the Aubin-Lions Lemma, we can claim $\rho_n \in C([0,T];H^3)$, so up to a subsequence, we have

$$\rho_n \rightarrow \rho \text{ strongly in } C([0,T];H^3), \text{ hence, } \rho_n \rightarrow \rho \text{ a.e.}$$

Next we claim that ρ_n^γ is bounded in $L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}})$.

Notice that $\nabla \rho_n^{\frac{\gamma}{2}}$ is bounded in $L^2(0,T;L^2)$, using the Sobolev embedding theorem gives us ρ_n^γ is bounded in $L^1(0,T;L^3)$, then we apply Hölder inequality to have

$$\|\rho_n^\gamma\|_{L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}})} \leq \|\rho_n^\gamma\|_{L^\infty(0,T;L^1)}^{\frac{2}{5}} \|\rho_n^\gamma\|_{L^1(0,T;L^3)}^{\frac{3}{5}} \leq K.$$

Similarly, we can show ρ_n^{-6} is bounded in $L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}})$ too. Moreover, for $\rho_n \rightarrow \rho$ a.e., so $\rho_n^\gamma \rightarrow \rho^\gamma$ a.e.. Recall that the pressure satisfies $\frac{1}{a}\rho^{\gamma-1} - b \leq P' \leq a\rho^{\gamma-1} + b$ and $P \in C^1(\mathbb{R}^+)$, integrating this inequality we have

$$|P(\rho_n)| \leq C(\rho_n^\gamma + \rho_n),$$

it implies that $P(\rho_n)$ is bounded in $L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}})$ due to ρ_n^γ is bounded in $L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}})$. For $\rho_n^\gamma \rightarrow \rho^\gamma$ a.e., using the Egoroffs theorem, we have

$$P(\rho_n) \rightarrow P(\rho) \text{ strongly in } L^1(0,T;L^1).$$

Next, we show that the density is bounded away from zero with a positive constant for all the time $t \in [0, T]$ by using the Sobolve inequality.

For

$$\|\rho_n^{-1}\|_{L^\infty} \leq \|\rho_n^{-1}\|_{L^6}^{\frac{1}{2}} \|\nabla^2 \rho_n^{-1}\|_{L^2}^{\frac{1}{2}}, \quad (3.28)$$

$$\begin{aligned} \|\nabla^2 \rho_n^{-1}\|_{L^2} &\leq C(\|\rho_n^{-2} \nabla^2 \rho_n\|_{L^2} + \|\rho_n^{-3} (\nabla \rho_n)^2\|_{L^2}) \\ &\leq C\|\rho_n^{-1}\|_{L^6}^2 (\|\rho_n\|_{L^1} + \|\nabla^3 \rho_n\|_{L^2}) + C\|\rho_n^{-1}\|_{L^6}^3 \|\nabla \rho_n\|_{L^\infty}^2 \\ &\leq C(1 + \|\rho_n^{-1}\|_{L^6}^3)(1 + \|\nabla^3 \rho_n\|_{L^2}^2), \end{aligned} \quad (3.29)$$

substituting (3.29) into (3.28), yields

$$\begin{aligned} \|\rho_n^{-1}\|_{L^\infty} &\leq C\|\rho_n^{-1}\|_{L^6}^{\frac{1}{2}} (1 + \|\rho_n^{-1}\|_{L^6})^{\frac{3}{2}} (1 + \|\nabla^3 \rho_n\|_{L^2}) \\ &\leq C(1 + \|\rho_n^{-1}\|_{L^6})^2 (1 + \|\nabla^3 \rho_n\|_{L^2}) \leq C(\eta, \delta, T), \end{aligned} \quad (3.30)$$

where here the constant $C(\eta, \delta, T)$ depends on η, δ and T but independent of n .

So immediately, we have $\frac{1}{\rho_n} \rightarrow \frac{1}{\rho}$ a.e., furthermore we show $\rho_n^{-6} \rightarrow \rho^{-6}$ a.e., together with $\rho_n^{-6} \in L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}})$ and Egoroffs theorem, we have

$$\rho_n^{-6} \rightarrow \rho^{-6}, \quad \text{strongly in } L^1(0, T; L^1).$$

By using the G-N inequality, yields

$$\|\nabla \Phi(\rho_n)\|_{L^2} \leq C\|\nabla^2 \Phi(\rho_n)\|_{L^{\frac{6}{5}}} \leq C\|\rho_n - 1\|_{L^{\frac{6}{5}}},$$

because ρ_n convergence to ρ strongly in $C([0, T]; H^3)$, so

$$\nabla \Phi(\rho_n) \rightarrow \nabla \Phi(\rho) \quad \text{strongly in } L^2(0, T; L^2).$$

Then the proof of this Lemma is completed. \square

3.3.2. Step2. Convergence of momentum.

Lemma 3.3. *Up to an extracted subsequence*

$$\rho_n u_n \rightarrow \rho u \quad \text{a.e. and strongly in } L^2(0, T; L^2).$$

Proof. From the energy estimates, we know that u_n is bounded in $L^2(0, T; L^2)$, so up to a subsequence, we have

$$u_n \rightharpoonup u \quad \text{in } L^2(0, T; L^2),$$

recall that $\rho_n \rightarrow \rho$ strongly in $C([0, T]; H^3)$, so we have

$$\rho_n u_n \rightharpoonup \rho u \quad \text{weakly in } L^1(0, T; L^1).$$

Moreover, since $\rho_n \in L^\infty(0, T; H^3)$, $u_n \in L^2(0, T; H^2)$, we can show

$$\nabla(\rho_n u_n) = \nabla \rho_n u_n + \rho_n \nabla u_n \in L^2(0, T; L^2),$$

together with $\rho_n u_n \in L^2(0, T; L^2)$, we have $\rho_n u_n \in L^2(0, T; H^1)$. Next in order to use the Aubin-Lions Lemma, we only need to prove

$$\partial_t(\rho_n u_n) \in L^2(0, T; H^{-s}), \quad \text{for some } s > 0.$$

Since,

$$\begin{aligned} \partial_t(\rho_n u_n) = & -\operatorname{div}(\rho_n u_n \otimes u_n) - \nabla P(\rho_n) + \eta \nabla \rho_n^{-6} - \mu \Delta^2 u_n + \operatorname{div}(\rho_n \mathbb{D} u_n) \\ & - r_0 u_n - r_1 \rho_n |u_n| u_n - \varepsilon \nabla \rho_n \cdot \nabla u_n + \delta \rho_n \nabla \Delta^3 \rho_n + \rho_n \nabla \Phi, \end{aligned} \quad (3.31)$$

based on the energy estimates, it's easy to check that $\partial_t(\rho_n u_n) \in L^2(0, T; H^{-3})$, then by using the Aubin-Lions Lemma, we can show

$$\rho_n u_n \rightarrow g \text{ strongly in } L^2(0, T; L^2), \text{ for some function } g \in L^2(0, T; L^2),$$

moreover, since $\rho_n u_n \rightharpoonup \rho u$ weakly in $L^1(0, T; L^1)$, so we have

$$\rho_n u_n \rightarrow \rho u \text{ strongly in } L^2(0, T; L^2).$$

Thus the proof of this Lemma is completed. \square

3.3.3. Step.3 Convergence of nonlinear diffusion terms. Let $\varphi \in C_{per}^\infty([0, T]; \mathbb{T}^3)$, with $C_{per}^\infty([0, T]; \mathbb{T}^3)$ defined by

$$C_{per}^\infty([0, T]; \mathbb{T}^3) = \{\phi \in C^\infty([0, T]; \mathbb{T}^3) \mid \phi \text{ is periodic in } x\}.$$

$$\begin{aligned} \int_0^T \int \operatorname{div}(\rho_n \mathbb{D} u_n) \varphi &= \int_0^T \int \partial_i (\rho_n (\frac{\partial_i u_n^j + \partial_j u_n^i}{2})) \varphi \\ &= -\frac{1}{2} \int_0^T \int \rho_n \partial_i u_n^j \partial_i \varphi - \frac{1}{2} \int_0^T \int \rho_n \partial_j u_n^i \partial_i \varphi \\ &= -\frac{1}{2} \int_0^T \int \partial_i (\rho_n u_n^j) \partial_i \varphi + \frac{1}{2} \int_0^T \int \partial_i \rho_n u_n^j \partial_i \varphi - \frac{1}{2} \int_0^T \int \partial_j (\rho_n u_n^i) \partial_i \varphi \\ &\quad + \frac{1}{2} \int_0^T \int \partial_j \rho_n u_n^i \partial_i \varphi \\ &= \frac{1}{2} \int_0^T \int (\rho_n u_n^j) \partial_{ii} \varphi + \frac{1}{2} \int_0^T \int \partial_i \rho_n u_n^j \partial_i \varphi + \frac{1}{2} \int_0^T \int (\rho_n u_n^i) \partial_{ij} \varphi \\ &\quad + \frac{1}{2} \int_0^T \int \partial_j \rho_n u_n^i \partial_i \varphi, \end{aligned} \quad (3.32)$$

since $\rho_n \rightarrow \rho$ strongly in $C([0, T]; H^3)$, $\rho_n u_n \rightarrow \rho u$ strongly in $L^2(0, T; L^2)$, $u_n \rightharpoonup u$ weakly in $L^2(0, T; L^2)$, so we have

$$\begin{aligned} \int_0^T \int (\rho_n u_n^j) \partial_{ii} \varphi &\rightarrow \int_0^T \int (\rho u^j) \partial_{ii} \varphi, \quad \int_0^T \int \partial_i \rho_n u_n^j \partial_i \varphi \rightarrow \int_0^T \int \partial_i \rho u^j \partial_i \varphi, \\ \int_0^T \int (\rho_n u_n^i) \partial_{ij} \varphi &\rightarrow \int_0^T \int (\rho u^i) \partial_{ij} \varphi, \quad \int_0^T \int \partial_j \rho_n u_n^i \partial_i \varphi \rightarrow \int_0^T \int \partial_j \rho u^i \partial_i \varphi. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_0^T \int \rho_n \nabla \Delta^3 \rho_n \varphi &= - \int_0^T \int (\rho_n \operatorname{div} \varphi + \varphi \cdot \nabla \rho_n) \Delta^3 \rho_n \\ &= - \int_0^T \int \Delta (\rho_n \operatorname{div} \varphi + \varphi \cdot \nabla \rho_n) \Delta^2 \rho_n, \end{aligned} \quad (3.33)$$

we focus on the most difficult term $-\int_0^T \int \varphi \cdot \nabla \Delta \rho_n \Delta^2 \rho_n$, because $\rho_n \rightarrow \rho$ strongly in $C([0, T]; H^3)$ and $\rho_n \rightharpoonup \rho$ in $L^2(0, T; H^4)$, we have

$$-\int_0^T \int \varphi \cdot \nabla \Delta \rho_n \Delta^2 \rho_n \rightarrow -\int_0^T \int \varphi \cdot \nabla \Delta \rho \Delta^2 \rho.$$

And we can apply the above arguments to handle the other terms from

$$-\int_0^T \int \Delta(\rho_n \operatorname{div} \varphi + \varphi \cdot \nabla \rho_n) \Delta^2 \rho_n.$$

Thus we have

$$\int_0^T \int \rho_n \nabla \Delta^3 \rho_n \varphi \rightarrow \int_0^T \int \rho \nabla \Delta^3 \rho \varphi, \quad \text{as } n \rightarrow \infty.$$

With the above compactness results in hand, we are ready to pass to the limits as $n \rightarrow \infty$ in the approximate system (3.2), (3.6). Thus, we can show that (ρ, u, Φ) solves

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= \varepsilon \Delta \rho, \quad \text{pointwise on } (0, T) \times \mathbb{T}^3, \\ \Delta \Phi &= -4\pi G(\rho - 1), \quad \text{pointwise on } (0, T) \times \mathbb{T}^3, \end{aligned}$$

and for any test function φ , the following holds

$$\begin{aligned} & \int \rho u(T) \varphi - \int m_0 \varphi + \mu \int_0^T \int \Delta u \cdot \Delta \varphi - \int_0^T \int (\rho u \otimes u) \cdot \nabla \varphi \\ & + \int_0^T \int \rho \mathbb{D} u \cdot \nabla \varphi - \int_0^T \int P(\rho) \operatorname{div} \varphi + \eta \int_0^T \int \rho^{-6} \operatorname{div} \varphi + \varepsilon \int_0^T \int \nabla \rho \cdot \nabla u \varphi \\ & + r_0 \int_0^T \int u \varphi + r_1 \int_0^T \int \rho |u| u \varphi - \delta \int_0^T \int \rho \nabla \Delta^3 \rho \varphi = \int_0^T \int \rho \Phi \varphi \end{aligned} \quad (3.34)$$

Thanks to the lower semicontinuity of norms, we can pass to the limits in the energy estimate (3.22), and we have the following energy inequality in the sense of distributions on $(0, T)$.

$$\begin{aligned} & \sup_{t \in (0, T)} E(t) + \frac{4\varepsilon}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 + \frac{2}{3} \eta \varepsilon \int_0^T \int |\nabla \rho^{-3}|^2 + \int_0^T \int \rho |\mathbb{D} u|^2 \\ & + r_0 \int_0^T \int |u|^2 + r_1 \int_0^T \int \rho |u|^3 + \mu \int_0^T \int |\Delta u|^2 + \delta \varepsilon \int_0^T \int |\Delta^2 \rho|^2 \\ & \leq C + C_\varepsilon, \end{aligned} \quad (3.35)$$

where

$$E(t) = \int \frac{1}{2} \rho u^2 + \frac{1}{a\gamma(\gamma-1)} \rho^\gamma + \frac{\eta}{7} \rho^{-6} + \frac{\delta}{2} |\nabla \Delta \rho|^2 dx,$$

and

$$C_\varepsilon = C\left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta}\right) \left[1 + \left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta}\right) T e^{C\left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta}\right) T}\right] T.$$

Thus, we have the following proposition on the existence of weak solutions at this level approximate system.

Proposition 3.3. *There exists a weak solution to the following system*

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = \varepsilon \Delta \rho, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - \eta \nabla \rho^{-6} - \operatorname{div}(\rho \mathbb{D} u) + \mu \Delta^2 u + \varepsilon \nabla \rho \cdot \nabla u \\ \quad r_0 u + r_1 \rho |u| u - \delta \rho \nabla \Delta^3 \rho = \rho \nabla \Phi, \\ \Delta \Phi = -4\pi G(\rho - 1), \end{cases} \quad (3.36)$$

with suitable initial data, for any $T > 0$. In particular, the weak solutions (ρ, u, Φ) satisfy the energy inequality (3.35) and (3.30).

4. B-D ENTROPY AND PASSING TO THE LIMITS AS $\varepsilon, \mu \rightarrow 0$

In this section, we deduce the B-D entropy estimate for the approximate system in Proposition 3.3 which was first introduced by Bresch and Desjardins in [2], this B-D entropy will give a higher regularity of the density and will help us to get the compactness of ρ . By (3.26), (3.30) and $u \in L^2(0, T; H^2)$, we have

$$\rho(x, t) \geq C(\delta, \eta) > 0, \quad \rho \in L^\infty(0, T; H^3) \cap L^2(0, T; H^4), \quad \text{and} \quad \partial_t \rho \in L^2(0, T; L^2). \quad (4.1)$$

4.1. B-D entropy. Thanks to (4.1), it's easy to check $\frac{\nabla \rho}{\rho} \in L^2(0, T; H^3)$ and $\partial_t \frac{\nabla \rho}{\rho} \in L^2(0, T; L^2)$, so we can use $\frac{\nabla \rho}{\rho}$ as a test function to test the momentum equation to derive the Bresch-Desjardins entropy. Thus, we have

Lemma 4.1.

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} \rho \left| \frac{\nabla \rho}{\rho} \right|^2 + \rho u \cdot \frac{\nabla \rho}{\rho} \right) dx + \delta \int |\Delta^2 \rho|^2 + \frac{2}{3} \eta \int |\nabla \rho^{-3}|^2 - \int \rho \partial_i u^j \partial_j u^i \\ & + \varepsilon \int \frac{1}{\rho} |\Delta \rho|^2 \\ & = \int \rho \nabla \Phi \cdot \frac{\nabla \rho}{\rho} - \int \nabla P(\rho) \cdot \frac{\nabla \rho}{\rho} - r_0 \int u \cdot \frac{\nabla \rho}{\rho} - r_1 \int \rho |u| u \cdot \frac{\nabla \rho}{\rho} \\ & - \mu \int \Delta u \cdot \Delta \left(\frac{\nabla \rho}{\rho} \right) - \varepsilon \int \operatorname{div}(\rho u) \frac{\Delta \rho}{\rho} - \varepsilon \int \partial_i \rho \partial_i u^j \frac{\partial_j \rho}{\rho} - \frac{\varepsilon}{2} \int \Delta \rho \left| \frac{\nabla \rho}{\rho} \right|^2 \\ & = \sum_{i=1}^8 I_i, \end{aligned} \quad (4.2)$$

firstly, we control the terms $I_1 - I_4$:

$$I_1 = \int \rho \nabla \Phi \cdot \frac{\nabla \rho}{\rho} = - \int \Delta \Phi (\rho - 1) = 4\pi G \int (\rho - 1)^2, \quad (4.3)$$

$$\begin{aligned} I_2 &= - \int \nabla P(\rho) \frac{\nabla \rho}{\rho} = - \int \frac{P'}{\rho} |\nabla \rho|^2 \\ &\leq - \int \frac{\frac{1}{a} \rho^{\gamma-1} - b}{\rho} |\nabla \rho|^2 = - \frac{4}{a\gamma^2} \int |\nabla \rho^{\frac{\gamma}{2}}|^2 + 4b \int |\nabla \sqrt{\rho}|^2, \end{aligned} \quad (4.4)$$

where we used the condition (1.3), besides this we can also have

$$\begin{aligned} I_3 &= -r_0 \int u \cdot \frac{\nabla \rho}{\rho} = -r_0 \int \rho^{-1} \operatorname{div}(\rho u) = -r_0 \int \rho^{-1} (\varepsilon \Delta \rho - \rho_t) \\ &= r_0 \int \partial_t \log \rho - r_0 \varepsilon \int \frac{1}{\rho^2} |\nabla \rho|^2, \end{aligned} \quad (4.5)$$

$$I_4 = -r_1 \int \rho |u| u \cdot \frac{\nabla \rho}{\rho} = r_1 \int \rho (|u| \operatorname{div} u + u \frac{u}{|u|} \nabla u) \leq C \|\sqrt{\rho} u\|_{L^2} \|\sqrt{\rho} \nabla u\|_{L^2}. \quad (4.6)$$

Substituting (4.3)-(4.6) into (4.2) and integrating it with respect to the time t over $[0, T]$, we have

$$\begin{aligned} & \frac{1}{2} \int \rho (u + \frac{\nabla \rho}{\rho})^2 - r_0 \log \rho \, dx + \varepsilon \int_0^T \int \frac{1}{\rho} |\Delta \rho|^2 + r_0 \varepsilon \int_0^T \int \rho^{-2} |\nabla \rho|^2 \\ & + \frac{2}{3} \eta \int_0^T \int |\nabla \rho^{-3}|^2 + \int_0^T \int \rho |\nabla u|^2 + \frac{4}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 + \delta \int_0^T \int |\Delta^2 \rho|^2 \\ & \leq \int \frac{1}{2} \rho |u|^2 + \int_0^T \int \rho |\mathbb{D} u|^2 + 4b \int_0^T \int |\nabla \sqrt{\rho}|^2 + 4\pi G \int_0^T \int (\rho - 1)^2 \\ & + C \int_0^T \|\sqrt{\rho} u\|_{L^2} \|\sqrt{\rho} \nabla u\|_{L^2} + \int_0^T \sum_{i=5}^8 I_i dt + \frac{1}{2} \int \rho_0 (u_0 + \frac{\nabla \rho_0}{\rho_0})^2 - r_0 \log \rho_0 \, dx \\ & \leq C + C_\varepsilon + \frac{1}{2} \int_0^T \|\sqrt{\rho} \nabla u\|_{L^2}^2 dt + 4b \int_0^T \int |\nabla \sqrt{\rho}|^2 + 4\pi G \int_0^T \int (\rho - 1)^2 \\ & + \int_0^T \sum_{i=5}^8 I_i dt, \end{aligned} \quad (4.7)$$

where we used the energy inequality (3.35). Then we need to control the rest terms on the right hand of the (4.7):

$$\begin{aligned} 4\pi G \int_0^T \int (\rho - 1)^2 &\leq C + C \int_0^T (\|\rho\|_{L^1}^{\frac{5\gamma-6}{2(5\gamma-3)}} \|\rho\|_{L^{\frac{5\gamma}{3}}}^{\frac{5\gamma}{2(5\gamma-3)}})^2 dt \\ &\leq C + C \int_0^T \|\rho\|_{L^1}^{\frac{5\gamma-6}{(5\gamma-3)}} (\|\rho^\gamma\|_{L^1}^{\frac{2}{5}} \|\rho^\gamma\|_{L^3}^{\frac{3}{5}})^{\frac{5}{5\gamma-3}} dt \\ &\leq C + C \int_0^T (C + C_\varepsilon)^{\frac{2}{5\gamma-3}} \|\rho^\gamma\|_{L^3}^{\frac{3}{5\gamma-3}} dt \\ &\leq \frac{1}{2} \frac{4}{a\gamma^2} \int_0^T \|\nabla \rho^{\frac{\gamma}{2}}\|_{L^2}^2 dt + (C + C_\varepsilon)^3, \end{aligned} \quad (4.8)$$

where we need to require $\frac{6}{5\gamma-3} < 2$, which implies $\gamma > \frac{6}{5}$.

$$\begin{aligned}
\int_0^T I_5 dt &= -\mu \int_0^T \int \Delta u \cdot \Delta \left(\frac{\nabla \rho}{\rho} \right) \\
&= -\mu \int_0^T \int \Delta u \left[\frac{1}{\rho} \operatorname{div} \nabla^2 \rho - \frac{\nabla \rho}{\rho^2} \nabla^2 \rho - \frac{1}{\rho^2} \operatorname{div} (\nabla \rho \otimes \nabla \rho) + 2 \nabla \rho \otimes \nabla \rho \frac{\nabla \rho}{\rho^3} \right] \\
&\leq C\mu \int_0^T \int |\Delta u| \left[\frac{1}{\rho} |\nabla^3 \rho| + \frac{1}{\rho^2} |\nabla \rho| |\nabla^2 \rho| + \frac{1}{\rho^3} |\nabla \rho|^3 \right] \\
&\leq C\sqrt{\mu} \|\sqrt{\mu} \Delta u\|_{L^2(L^2)} \left[\left\| \frac{1}{\rho} \right\|_{L^\infty(L^\infty)} \|\nabla^3 \rho\|_{L^2(L^2)} + \left\| \frac{1}{\rho} \right\|_{L^\infty(L^\infty)}^2 \|\nabla \rho\|_{L^\infty(L^\infty)} \|\nabla^2 \rho\|_{L^2(L^2)} \right. \\
&\quad \left. + \left\| \frac{1}{\rho} \right\|_{L^\infty(L^\infty)}^3 \|\nabla \rho\|_{L^6(L^6)}^3 \right] \\
&\leq C\sqrt{\mu} \|\sqrt{\mu} \Delta u\|_{L^2(L^2)} \left[\left\| \frac{1}{\rho} \right\|_{L^\infty(L^\infty)}^3 \|\nabla^3 \rho\|_{L^2(L^2)}^3 + \left\| \frac{1}{\rho} \right\|_{L^\infty(L^\infty)} \right] \\
&\leq C(\delta, \eta)(C + C_\varepsilon)^s \sqrt{\mu},
\end{aligned} \tag{4.9}$$

for some large fixed constant $s > 0$.

$$\begin{aligned}
\int_0^T I_6 dt &= -\varepsilon \int_0^T \int \operatorname{div}(\rho u) \frac{\Delta \rho}{\rho} = -\varepsilon \int_0^T \int \operatorname{div} u \Delta \rho - \varepsilon \int_0^T \int \frac{u \cdot \nabla \rho}{\rho} \Delta \rho \\
&\leq C\varepsilon \|\nabla u\|_{L^2(L^6)} \|\Delta \rho\|_{L^2(L^{\frac{6}{5}})} + C\varepsilon \|\rho^{\frac{1}{3}} u\|_{L^3(L^3)} \|\rho^{-\frac{4}{3}}\|_{L^\infty(L^\infty)} \|\nabla \rho\|_{L^6(L^6)} \|\Delta \rho\|_{L^2(L^2)} \\
&\leq C\sqrt{\varepsilon} (\|\sqrt{\varepsilon} \nabla^2 u\|_{L^2(L^2)} + \|u\|_{L^2(L^2)}) (\|\rho\|_{L^\infty(L^1)} + \|\nabla^3 \rho\|_{L^\infty(L^2)}) \\
&\quad + C\varepsilon \|\rho^{\frac{1}{3}} u\|_{L^3(L^3)} \|\rho^{-\frac{4}{3}}\|_{L^\infty(L^\infty)} (\|\rho\|_{L^\infty(L^1)} + \|\nabla^3 \rho\|_{L^\infty(L^2)})^2 \\
&\leq C(\delta, \eta, r_0)(C + C_\varepsilon)^s (\sqrt{\varepsilon} + \varepsilon),
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\int_0^T I_7 dt &= -\varepsilon \int \partial_i \rho \partial_i u^j \frac{\partial_j \rho}{\rho} \leq C\sqrt{\varepsilon} \|\rho^{-1}\|_{L^\infty(L^\infty)} \|\nabla \rho\|_{L^4(L^4)}^2 \|\nabla u\|_{L^2(L^2)} \\
&\leq C(\delta, \eta) \sqrt{\varepsilon} \|\rho^{-1}\|_{L^\infty(L^\infty)} \|\nabla^3 \rho\|_{L^\infty(L^2)}^2 \|\sqrt{\varepsilon} \nabla^2 u\|_{L^2(L^2)} \\
&\leq C(\delta, \eta, r_0)(C + C_\varepsilon)^s \sqrt{\varepsilon},
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
\int_0^T I_8 dt &= -\frac{\varepsilon}{2} \int \Delta \rho \left| \frac{\nabla \rho}{\rho} \right|^2 \leq C\varepsilon \int_0^T \|\rho^{-1}\|_{L^\infty}^2 \|\Delta \rho\|_{L^2} \|\nabla \rho\|_{L^4}^2 dt \\
&\leq C\varepsilon \|\rho^{-1}\|_{L^\infty(L^\infty)}^2 (\|\nabla^3 \rho\|_{L^\infty(L^2)} + \|\rho\|_{L^\infty(L^1)})^3 \\
&\leq C(\delta, \eta, r_0)(C + C_\varepsilon)^s \varepsilon.
\end{aligned} \tag{4.12}$$

Then substituting (4.8)-(4.12) into (4.7), we have

$$\begin{aligned}
& \frac{1}{2} \int \rho(u + \frac{\nabla \rho}{\rho})^2 - r_0 \log \rho dx + \varepsilon \int_0^T \int \frac{1}{\rho} |\Delta \rho|^2 + r_0 \varepsilon \int_0^T \int \rho^{-2} |\nabla \rho|^2 \\
& + \frac{2}{3} \eta \int_0^T \int |\nabla \rho^{-3}|^2 + \frac{1}{2} \int_0^T \int \rho |\nabla u|^2 + \frac{2}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 + \delta \int_0^T \int |\Delta^2 \rho|^2 \\
& \leq 4b \int_0^T \int |\nabla \sqrt{\rho}|^2 + C(\delta, \eta, r_0)(C + C_\varepsilon)^s (\varepsilon + \sqrt{\varepsilon} + \sqrt{\mu}) + (C + C_\varepsilon)^3,
\end{aligned} \tag{4.13}$$

by using the Gronwall inequality, yields

$$\int |\nabla \sqrt{\rho}|^2 dx \leq [C(\delta, \eta, r_0)(C + C_\varepsilon)^s (\varepsilon + \sqrt{\varepsilon} + \sqrt{\mu}) + (C + C_\varepsilon)^3](1 + 4bT \exp^{4bT}), \tag{4.14}$$

so with this inequality and (4.13), we have the following B-D entropy estimats

$$\begin{aligned}
& \frac{1}{2} \int \rho(u + \frac{\nabla \rho}{\rho})^2 - r_0 \log \rho dx + \varepsilon \int_0^T \int |\Delta \rho|^2 + r_0 \varepsilon \int_0^T \int \rho^{-2} |\nabla \rho|^2 \\
& + \frac{2}{3} \eta \int_0^T \int |\nabla \rho^{-3}|^2 + \int_0^T \int \rho |\nabla u|^2 + \frac{4}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 + \delta \int_0^T \int |\Delta^2 \rho|^2 \\
& \leq [C(\delta, \eta, r_0)(C + C_\varepsilon)^s (\varepsilon + \sqrt{\varepsilon} + \sqrt{\mu}) + (C + C_\varepsilon)^3][(1 + 4bT \exp^{4bT}) + 1],
\end{aligned} \tag{4.15}$$

where $s > 0$ is a suitable large fixed constant, C is a generic positive constant depending on the initial data and other constants but independent of $\varepsilon, \delta, \eta, r_0$, and $C_\varepsilon = C(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta})[1 + (\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta})T e^{C(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta})T}]T$.

Thus, at this approximation level, with δ, η, r_0 being fixed and $\varepsilon \ll 1, \mu \ll 1$, from (3.35) and (4.15) we get the following energy inequality and the B-D entropy

$$\begin{aligned}
& \sup_{t \in (0, T)} \int \frac{1}{2} \rho_{\mu, \varepsilon} u_{\mu, \varepsilon}^2 + \frac{1}{a\gamma(\gamma - 1)} \rho_{\mu, \varepsilon}^\gamma + \frac{\eta}{7} \rho_{\mu, \varepsilon}^{-6} + \frac{\delta}{2} |\nabla \Delta \rho_{\mu, \varepsilon}|^2 dx + \frac{4\varepsilon}{a\gamma^2} \int_0^T \int |\nabla \rho_{\mu, \varepsilon}^{\frac{\gamma}{2}}|^2 \\
& + \frac{2}{3} \eta \varepsilon \int_0^T \int |\nabla \rho_{\mu, \varepsilon}^{-3}|^2 + \int_0^T \int \rho_{\mu, \varepsilon} |\mathbb{D} u_{\mu, \varepsilon}|^2 + r_0 \int_0^T \int |u_{\mu, \varepsilon}|^2 + r_1 \int_0^T \int \rho_{\mu, \varepsilon} |u_{\mu, \varepsilon}|^3 \\
& + \mu \int_0^T \int |\Delta u_{\mu, \varepsilon}|^2 + \delta \varepsilon \int_0^T \int |\Delta^2 \rho_{\mu, \varepsilon}|^2 \leq C(\delta, \eta, T),
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
& \frac{1}{2} \int \rho_{\mu, \varepsilon} (u_{\mu, \varepsilon} + \frac{\nabla \rho_{\mu, \varepsilon}}{\rho_{\mu, \varepsilon}})^2 - r_0 \log \rho_{\mu, \varepsilon} dx + \varepsilon \int_0^T \int \frac{1}{\rho_{\mu, \varepsilon}} |\Delta \rho_{\mu, \varepsilon}|^2 + r_0 \varepsilon \int_0^T \int \rho_{\mu, \varepsilon}^{-2} |\nabla \rho_{\mu, \varepsilon}|^2 \\
& + \frac{2}{3} \eta \int_0^T \int |\nabla \rho_{\mu, \varepsilon}^{-3}|^2 + \frac{1}{2} \int_0^T \int \rho_{\mu, \varepsilon} |\nabla u|^2 + \frac{2}{a\gamma^2} \int_0^T \int |\nabla \rho_{\mu, \varepsilon}^{\frac{\gamma}{2}}|^2 + \delta \int_0^T \int |\Delta^2 \rho_{\mu, \varepsilon}|^2 \\
& \leq C(\delta, \eta, T),
\end{aligned} \tag{4.17}$$

where $C(\delta, \eta, T)$ denotes that C particularly depends on δ, η and time T .

4.2. Passing to the limits as $\mu, \varepsilon \rightarrow 0$. We use $(\rho_{\mu,\varepsilon}, u_{\mu,\varepsilon}, \Phi(\rho_{\mu,\varepsilon}))$ to denote the solutions at this level of approximation. From (4.16), (4.17), it's easy to show that $(\rho_{\mu,\varepsilon}, u_{\mu,\varepsilon}, \Phi(\rho_{\mu,\varepsilon}))$ has the following uniform regularities

$$\begin{aligned} \sqrt{\rho_{\mu,\varepsilon}} u_{\mu,\varepsilon} &\in L^\infty(0, T; L^2), \sqrt{\rho_{\mu,\varepsilon}} \mathbb{D} u_{\mu,\varepsilon} \in L^2(0, T; L^2), \\ \rho_{\mu,\varepsilon}^\gamma &\in L^\infty(0, T; L^1), \sqrt{\varepsilon} \nabla \rho_{\mu,\varepsilon}^{\frac{\gamma}{2}} \in L^2(0, T; L^2), \eta \rho_{\mu,\varepsilon}^{-6} \in L^\infty(0, T; L^1), \\ \sqrt{\varepsilon} \nabla \rho_{\mu,\varepsilon}^{-3} &\in L^2(0, T; L^2), \rho_{\mu,\varepsilon} \in L^\infty(0, T; H^3), \sqrt{\varepsilon} \rho_{\mu,\varepsilon} \in L^2(0, T; H^4), \\ u_{\mu,\varepsilon} &\in L^2(0, T; L^2), \rho_{\mu,\varepsilon}^{\frac{1}{3}} u_{\mu,\varepsilon} \in L^3(0, T; L^3), \sqrt{\mu} \Delta u_{\mu,\varepsilon} \in L^2(0, T; L^2), \end{aligned} \quad (4.18)$$

by the Bresch-Dejardins entropy, we also have the following additional regularities

$$\begin{aligned} \nabla \sqrt{\rho_{\mu,\varepsilon}} &\in L^\infty(0, T; L^2), \sqrt{\delta} \rho_{\mu,\varepsilon} \in L^2(0, T; H^4), \nabla \rho_{\mu,\varepsilon}^{\frac{\gamma}{2}} \in L^2(0, T; L^2) \\ \sqrt{\eta} \nabla \rho_{\mu,\varepsilon}^{-3} &\in L^2(0, T; L^2), \sqrt{\rho_{\mu,\varepsilon}} \nabla u_{\mu,\varepsilon} \in L^2(0, T; L^2). \end{aligned} \quad (4.19)$$

Based on the above regularities, we have the following uniform compactness results:

Lemma 4.2. *Let $(\rho_{\mu,\varepsilon}, u_{\mu,\varepsilon}, \Phi(\rho_{\mu,\varepsilon}))$ be weak solutions to (3.36), in combination with (4.18) and (4.19), we have*

$$\begin{aligned} \rho_{\mu,\varepsilon} &\in L^2(0, T; H^4), \quad \partial_t \rho_{\mu,\varepsilon} \in L^2(0, T; H^{-1}), \\ \rho_{\mu,\varepsilon} u_{\mu,\varepsilon} &\in L^2(0, T; W^{1, \frac{3}{2}}), \quad \partial_t (\rho_{\mu,\varepsilon} u_{\mu,\varepsilon}) \in L^2(0, T; H^{-3}), \\ \rho_{\mu,\varepsilon}^\gamma &\in L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}}), \quad \rho_{\mu,\varepsilon}^{-6} \in L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}}), \\ P(\rho_{\mu,\varepsilon}) &\in L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}}), \quad \Phi(\rho_{\mu,\varepsilon}) \in L^\infty(0, T; H^2), \end{aligned} \quad (4.20)$$

and using Aubin-Lions Lemma, we have the following compactness results

$$\begin{aligned} \rho_{\mu,\varepsilon} &\rightarrow \rho, \text{ a.e. and strongly in } C([0, T]; H^3), \quad \rho_{\mu,\varepsilon} \rightharpoonup \rho, \text{ in } L^2(0, T; H^4), \\ P(\rho_{\mu,\varepsilon}) &\rightarrow P(\rho), \text{ a.e. and strongly in } L^1(0, T; L^1), \\ u_{\mu,\varepsilon} &\rightharpoonup u, \text{ in } L^2(0, T; L^2), \\ \rho_{\mu,\varepsilon} u_{\mu,\varepsilon} &\rightarrow \rho u \text{ strongly in } L^2(0, T; L^p), \text{ for } \forall 1 \leq p < 3, \\ \rho_{\mu,\varepsilon}^{-6} &\rightarrow \rho^{-6}, \text{ a.e. and strongly in } L^1(0, T; L^1), \\ \nabla \Phi(\rho_{\mu,\varepsilon}) &\rightarrow \nabla \Phi(\rho) \text{ strongly in } L^2(0, T; L^2), \\ -\rho_{\mu,\varepsilon} \mathbb{D} u_{\mu,\varepsilon} &\rightarrow -\rho \mathbb{D} u, \text{ in the sense of distribution on } (0, T) \times \mathbb{T}^3, \\ -\delta \rho_{\mu,\varepsilon} \nabla \Delta^3 \rho_{\mu,\varepsilon} &\rightarrow -\delta \rho \nabla \Delta^3 \rho, \text{ in the sense of distribution on } (0, T) \times \mathbb{T}^3, \end{aligned} \quad (4.21)$$

Proof. The proof is similar to the compactness analysis in section 2, we repeat the compactness arguments here again and for the simplicity, we omit the details here. \square

With the above compactness results in hand, then we pass to the limits as $\mu = \varepsilon \rightarrow 0$, here we only focus on the terms involving with ε and μ . Firstly, because $\rho_{\mu,\varepsilon}$

is bounded in $L^\infty(H^3) \cap L^2(H^4)$ uniformly on ε , we have

$$\varepsilon \int_0^T \int \Delta \rho_{\mu,\varepsilon} \varphi \leq \varepsilon \|\Delta \rho_{\mu,\varepsilon}\|_{L^2(L^2)} \|\varphi\|_{L^2(L^2)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \quad (4.22)$$

for any test function $\varphi \in C_{per}^\infty([0, T]; \mathbb{T}^3)$. So passing to the limits in (3.36)₁ and using the Lemma 4.2, we have

$$\rho_t + \operatorname{div}(\rho u) = 0, \quad \text{a.e. in } (0, T) \times \mathbb{T}^3. \quad (4.23)$$

Similarly,

$$\begin{aligned} \varepsilon \int_0^T \int \nabla \rho_{\mu,\varepsilon} \cdot \nabla u_{\mu,\varepsilon} \varphi \\ \leq \sqrt{\varepsilon} \|\nabla \rho_{\mu,\varepsilon}\|_{L^2(L^2)} \|\sqrt{\varepsilon} \nabla u_{\mu,\varepsilon}\|_{L^2(L^2)} \|\varphi\|_{L^\infty(L^\infty)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} \mu \int_0^T \int \Delta^2 u_{\mu,\varepsilon} \varphi &= \mu \int_0^T \int \Delta u_{\mu,\varepsilon} \Delta \varphi \\ &\leq \sqrt{\mu} \|\sqrt{\mu} \Delta u_{\mu,\varepsilon}\|_{L^2(L^2)} \|\Delta \varphi\|_{L^2(L^2)} \rightarrow 0, \quad \text{as } \mu \rightarrow 0. \end{aligned} \quad (4.25)$$

So pass to the limits as $\mu = \varepsilon \rightarrow 0$ in (3.36), we have

$$\begin{aligned} (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - \eta \nabla \rho^{-6} - \operatorname{div}(\rho \mathbb{D} u) + r_0 u + r_1 \rho |u| u \\ - \delta \rho \nabla \Delta^3 \rho = \rho \nabla \Phi, \quad \text{holds in the sense of distribution on } (0, T) \times \mathbb{T}^3, \end{aligned} \quad (4.26)$$

and

$$\Delta \Phi = -4\pi G(\rho - 1), \quad \text{holds a.e. on } (0, T) \times \mathbb{T}^3. \quad (4.27)$$

Furthermore, thanks to lower semi-continuity of the convex function and the strong convergence of $\rho_{\mu,\varepsilon}, u_{\mu,\varepsilon}, \Phi(\rho_{\mu,\varepsilon})$, we can pass to the limits in the energy inequality (3.35) and Bresch-Desjardins entropy (4.15) as $\mu = \varepsilon \rightarrow 0$ with δ, η, r_0 being fixed,

$$\begin{aligned} \int \frac{1}{2} \rho u^2 + \frac{1}{a\gamma(\gamma-1)} \rho^\gamma + \frac{\eta}{7} \rho^{-6} + \frac{\delta}{2} |\nabla \Delta \rho|^2 dx + \int_0^T \int \rho |\mathbb{D} u|^2 \\ + r_0 \int_0^T \int |u|^2 + r_1 \int_0^T \int \rho |u|^3 \leq C(T), \end{aligned} \quad (4.28)$$

and

$$\begin{aligned} \frac{1}{2} \int \rho (u + \frac{\nabla \rho}{\rho})^2 - r_0 \log \rho dx + \frac{2}{3} \eta \int_0^T \int |\nabla \rho^{-3}|^2 + \frac{1}{2} \int_0^T \int \rho |\nabla u|^2 \\ + \frac{2}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 + \delta \int_0^T \int |\Delta^2 \rho|^2 \leq C(T), \end{aligned} \quad (4.29)$$

where $C(T)$ is a generic positive constant independent of $\varepsilon, \eta, \delta, r_0$, and we used the fact that $C_\varepsilon = C(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta})[1 + (\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta})T e^{C(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{\eta})T}]T \rightarrow 0$, as $\varepsilon \rightarrow 0$.

Thus, to conclude this part, we have the following proposition

Proposition 4.1. *There exists the weak solutions to the system (4.23), (4.26) and (4.27) with suitable initial data, for any $T > 0$. In particular, the weak solutions (ρ, u, Φ) satisfy the energy inequality (4.28) and the B-D entropy (4.29).*

5. PASSING TO THE LIMITS AS $\eta \rightarrow 0$.

In this section, we pass to the limits as $\eta \rightarrow 0$ with δ, r_0 being fixed. we denote that $(\rho_\eta, u_\eta, \Phi(\rho_\eta))$ are weak solutions at this level, from the proposition 4.1, we have the following regularities

$$\begin{aligned} \sqrt{\rho_\eta} u_\eta &\in L^\infty(0, T; L^2), \quad \sqrt{\rho_\eta} \mathbb{D} u_\eta \in L^2(0, T; L^2), \\ \nabla \sqrt{\rho_\eta} &\in L^\infty(0, T; L^2), \quad \sqrt{\delta} \rho_\eta \in L^\infty(0, T; H^3), \quad \sqrt{\delta} \rho_\eta \in L^2(0, T; H^4), \\ u_\eta &\in L^2(0, T; L^2), \quad \rho_\eta^{\frac{1}{3}} u_\eta \in L^3(0, T; L^3), \quad \sqrt{\rho_\eta} \nabla u_\eta \in L^2(0, T; L^2), \\ \rho_\eta^\gamma &\in L^\infty(0, T; L^1), \quad \nabla \rho_\eta^{\frac{\gamma}{2}} \in L^2(0, T; L^2), \\ \eta \rho_\eta^{-6} &\in L^\infty(0, T; L^1), \quad \sqrt{\eta} \nabla \rho_\eta^{-3} \in L^2(0, T; L^2). \end{aligned} \tag{5.1}$$

So it's easy to check that we have the same estimates as in Lemma 4.2 at the level with η , thus we deduce the same compactness for $(\rho_\eta, u_\eta, \Phi(\rho_\eta))$ as follows

$$\begin{aligned} \rho_\eta &\rightarrow \rho, \text{ a.e. and strongly in } C([0, T]; H^3), \quad \rho_\eta \rightharpoonup \rho, \text{ in } L^2(0, T; H^4), \\ P(\rho_\eta) &\rightarrow P(\rho), \text{ a.e. and strongly in } L^1(0, T; L^1), \\ u_\eta &\rightharpoonup u, \text{ in } L^2(0, T; L^2), \\ \rho_\eta u_\eta &\rightarrow \rho u \text{ a.e. and strongly in } L^2(0, T; L^2), \\ \nabla \Phi(\rho_\eta) &\rightarrow \nabla \Phi(\rho) \text{ strongly in } L^2(0, T; L^2), \\ -\rho_\eta \mathbb{D} u_\eta &\rightarrow -\rho \mathbb{D} u, \text{ in the sense of distribution on } (0, T) \times \mathbb{T}^3, \\ -\delta \rho_\eta \nabla \Delta^3 \rho_\eta &\rightarrow -\delta \rho \nabla \Delta^3 \rho, \text{ in the sense of distribution on } (0, T) \times \mathbb{T}^3. \end{aligned} \tag{5.2}$$

So at this level of approximation, we only focus on the convergence of the term $\eta \nabla \rho_\eta^{-6}$. Here we state the following Lemma.

Lemma 5.1. *For ρ_η defined as in Proposition 4.1, we have*

$$\eta \int_0^T \int \rho_\eta^{-6} dx dt \rightarrow 0$$

as $\eta \rightarrow 0$.

Proof. The proof is inspired by Vasseur and Yu in [22]. From the B-D entropy (4.29), we have

$$\sup_{t \in [0, T]} \int (\log(\frac{1}{\rho_\eta}))_+ dx \leq C(r_0) < +\infty. \tag{5.3}$$

Note that

$$y \in \mathbb{R}^+ \rightarrow \log(\frac{1}{y})_+$$

is a convex continuous function. Moreover, in combination with the property of the convex function and Fatou's Lemma, yields

$$\begin{aligned} \int (\log(\frac{1}{\rho}))_+ dx &\leq \int \liminf_{\eta \rightarrow 0} (\log(\frac{1}{\rho_\eta}))_+ dx \\ &\leq \liminf_{\eta \rightarrow 0} \int (\log(\frac{1}{\rho_\eta}))_+ dx, \end{aligned} \quad (5.4)$$

which implies $(\log(\frac{1}{\rho}))_+$ is bounded in $L^\infty(0, T; L^1)$, so it allows us to deduce that

$$|\{x \mid \rho(t, x) = 0\}| = 0, \quad \text{for almost every } t \in [0, T], \quad (5.5)$$

where $|A|$ denotes the measure of set A .

Thanks to the compactness of the density: $\rho_\eta \rightarrow \rho$ strongly in $C([0, T]; H^3)$, hence $\rho_\eta \rightarrow \rho$ a.e., then together with (5.5), we deduce

$$\eta \rho_\eta^{-6} \rightarrow 0 \quad \text{a.e.} \quad (5.6)$$

Moreover, using the interpolation inequality, yields

$$\|\eta \rho_\eta^{-6}\|_{L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}})} \leq \|\eta \rho_\eta^{-6}\|_{L^\infty(0, T; L^1)}^{\frac{2}{5}} \|\eta \rho_\eta^{-6}\|_{L^1(0, T; L^3)}^{\frac{3}{5}} \leq C,$$

this together with (5.6) and using the Eogroffs theorem, yields

$$\eta \rho_\eta^{-6} \rightarrow 0, \quad \text{strongly in } L^1(0, T; L^1).$$

□

Thus, by using the compactness results (5.2), we can pass to the limit as $\eta \rightarrow 0$ in (4.23), (4.26) and (4.27), yields

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= 0, \quad \text{holds a.e. on } (0, T) \times \mathbb{T}^3, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - \operatorname{div}(\rho \mathbb{D}u) + r_0 u + r_1 \rho |u| u \\ &\quad - \delta \rho \nabla \Delta^3 \rho = \rho \nabla \Phi, \quad \text{holds in the sense of distribution on } (0, T) \times \mathbb{T}^3, \\ \Delta \Phi &= -4\pi G(\rho - 1), \quad \text{holds a.e. on } (0, T) \times \mathbb{T}^3. \end{aligned} \quad (5.7)$$

Similarly, due to the lower semi-continuity of convex functions, we can obtain the energy inequality and B-D entropy by passing to the limits in (4.28) and (4.29) as $\eta \rightarrow 0$, we have

$$\begin{aligned} \int \frac{1}{2} \rho u^2 + \frac{1}{a\gamma(\gamma-1)} \rho^\gamma + \frac{\delta}{2} |\nabla \Delta \rho|^2 dx + \int_0^T \int \rho |\mathbb{D}u|^2 \\ + r_0 \int_0^T \int |u|^2 + r_1 \int_0^T \int \rho |u|^3 \leq C(T), \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} \frac{1}{2} \int \rho (u + \frac{\nabla \rho}{\rho})^2 - r_0 \log \rho dx + \frac{1}{2} \int_0^T \int \rho |\nabla u|^2 \\ + \frac{2}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 + \delta \int_0^T \int |\Delta^2 \rho|^2 \leq C(T), \end{aligned} \quad (5.9)$$

Thus we have the following Proposition on the existence of the weak solutions at this level of approximation.

Proposition 5.1. *There exist weak solutions to the system (5.7) with suitable initial data, for any $T > 0$. In particular, the weak solutions $(\rho, u, \Phi(\rho))$ satisfy the energy inequality (5.8) and the B-D entropy (5.9).*

6. PASSING TO THE LIMITS AS $\delta, r_0 \rightarrow 0$.

At this level, the weak solutions satisfy the energy inequality (5.8) and the B-D entropy (5.9), thus we have the following regularities:

$$\begin{aligned} \sqrt{\rho_{\delta, r_0}} u_{\delta, r_0} &\in L^\infty(0, T; L^2), \quad \sqrt{\rho_{\delta, r_0}} \mathbb{D} u_{\delta, r_0} \in L^2(0, T; L^2), \\ \nabla \sqrt{\rho_{\delta, r_0}} &\in L^\infty(0, T; L^2), \quad \sqrt{\delta} \rho_{\delta, r_0} \in L^\infty(0, T; H^3) \cap L^2(0, T; H^4), \\ \rho_{\delta, r_0}^\gamma &\in L^\infty(0, T; L^1), \quad \nabla \rho_{\delta, r_0}^{\frac{\gamma}{2}} \in L^2(0, T; L^2), \quad \sqrt{\rho_{\delta, r_0}} \nabla u_{\delta, r_0} \in L^2(0, T; L^2), \\ \sqrt{r_0} u_{\delta, r_0} &\in L^2(0, T; L^2), \quad \rho_{\delta, r_0}^{\frac{1}{3}} u_{\delta, r_0} \in L^3(0, T; L^3), \end{aligned} \tag{6.1}$$

Next, we will proceed the compactness arguments in several steps

6.1. Step 1: Convergence of $\sqrt{\rho_{\delta, r_0}}$.

Lemma 6.1. *Let $(\rho_{\delta, r_0}, u_{\delta, r_0}, \Phi(\rho_{\delta, r_0}))$ satisfy the Proposition 5.1, we have*

$$\begin{aligned} \sqrt{\rho_{\delta, r_0}} &\text{is bounded in } L^\infty(0, T; H^1), \\ \partial_t \sqrt{\rho_{\delta, r_0}} &\text{is bounded in } L^2(0, T; H^{-1}). \end{aligned}$$

As a consequence, up to a subsequence, $\sqrt{\rho_{\delta, r_0}}$ convergences almost everywhere and strongly in $L^2(0, T; L^2)$, which means

$$\sqrt{\rho_{\delta, r_0}} \rightarrow \sqrt{\rho}, \quad \text{a.e. and strongly in } L^2(0, T; L^2).$$

Moreover, we have

$$\rho_{\delta, r_0} \rightarrow \rho \quad \text{a.e. and strongly in } C([0, T]; L^p), \quad \text{for any } p \in [1, 3).$$

Proof. In combination with the conservation of the mass $\|\rho_{\delta, r_0}(t)\|_{L^1} = \|\rho_{\delta, r_0}(0)\|_{L^1}$ and estimate in (6.1) gives $\sqrt{\rho_{\delta, r_0}} \in L^\infty(0, T; H^1)$. Next, we notice that

$$\begin{aligned} \partial_t \sqrt{\rho_{\delta, r_0}} &= -\frac{1}{2} \sqrt{\rho_{\delta, r_0}} \operatorname{div} u_{\delta, r_0} - u_{\delta, r_0} \nabla \sqrt{\rho_{\delta, r_0}} \\ &= \frac{1}{2} \sqrt{\rho_{\delta, r_0}} \operatorname{div} u_{\delta, r_0} - \operatorname{div}(u_{\delta, r_0} \sqrt{\rho_{\delta, r_0}}), \end{aligned} \tag{6.2}$$

which yields $\partial_t \sqrt{\rho_{\delta, r_0}} \in L^2(0, T; H^{-1})$, thanks to the Aubin-Lions Lemma, we have

$$\sqrt{\rho_{\delta, r_0}} \rightarrow \sqrt{\rho}, \quad \text{strongly in } L^2(0, T; L^2),$$

and hence yields $\sqrt{\rho_{\delta, r_0}} \rightarrow \sqrt{\rho}$ a.e.. In another hand, since $\nabla \sqrt{\rho_{\delta, r_0}}$ is bounded in $L^\infty(0, T; L^2)$, by using the Sobolev embedding theorem, we have

$$\|\sqrt{\rho_{\delta, r_0}}\|_{L^\infty(L^6)} \leq C \|\nabla \sqrt{\rho_{\delta, r_0}}\|_{L^\infty(L^2)} < +\infty,$$

so

$$\rho_{\delta,r_0} u_{\delta,r_0} = \sqrt{\rho_{\delta,r_0}} \sqrt{\rho_{\delta,r_0}} u_{\delta,r_0} \in L^\infty(0, T; L^{\frac{3}{2}}), \quad (6.3)$$

which yields that $\operatorname{div}(\rho_{\delta,r_0} u_{\delta,r_0}) \in L^\infty(0, T; W^{-1, \frac{3}{2}})$, so the continuity equation yields $\partial_t \rho_{\delta,r_0} \in L^\infty(0, T; W^{-1, \frac{3}{2}})$. Furthermore, because

$$\nabla \rho_{\delta,r_0} = 2\sqrt{\rho_{\delta,r_0}} \nabla \sqrt{\rho_{\delta,r_0}} \in L^\infty(0, T; L^{\frac{3}{2}}), \quad (6.4)$$

so we have ρ_{δ,r_0} is bounded in $L^\infty(0, T; W^{1, \frac{3}{2}})$.

Then together with $\partial_t \rho_{\delta,r_0} \in L^\infty(0, T; W^{-1, \frac{3}{2}})$, thanks to the Aubin-Lions Lemma gives

$$\rho_{\delta,r_0} \rightarrow \rho, \text{ strongly in } C([0, T]; L^p), \text{ for any } p \in [1, 3), \quad (6.5)$$

and hence, we have

$$\rho_{\delta,r_0} \rightarrow \rho \text{ a.e..}$$

Thus the proof of this Lemma is completed. \square

6.2. Step 2: Convergence of the pressure.

Lemma 6.2. *The pressure $P(\rho_{\delta,r_0})$ satisfies the following regularity:*

$$P(\rho_{\delta,r_0}) \in L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}}),$$

and up to subsequence, we have

$$P(\rho_{\delta,r_0}) \rightarrow P(\rho) \text{ a.e.,}$$

and

$$P(\rho_{\delta,r_0}) \rightarrow P(\rho) \text{ strongly in } L^1(0, T; L^1).$$

Proof. The proof is as the same as it in section 2, so we omit the details here. \square

6.3. Step 3: Convergence of the momentum.

Lemma 6.3. *Up to a subsequence, the momentum $m_{\delta,r_0} = \rho_{\delta,r_0} u_{\delta,r_0}$ converges strongly in $L^2(0, T; L^q)$ to some $m(x, t)$ for all $q \in [1, \frac{3}{2})$. In particular*

$$\rho_{\delta,r_0} u_{\delta,r_0} \rightarrow m \text{ a.e. for } (x, t) \in \mathbb{T}^3 \times (0, T).$$

Note that we can define $u(x, t) = m(x, t)/\rho(x, t)$ outside the vacuum set $\{x | \rho(x, t) = 0\}$.

Proof. Since

$$\begin{aligned} \nabla(\rho_{\delta,r_0} u_{\delta,r_0}) &= \nabla \rho_{\delta,r_0} u_{\delta,r_0} + \rho_{\delta,r_0} \nabla u_{\delta,r_0} \\ &= 2\nabla \sqrt{\rho_{\delta,r_0}} \sqrt{\rho_{\delta,r_0}} u_{\delta,r_0} + \sqrt{\rho_{\delta,r_0}} \sqrt{\rho_{\delta,r_0}} \nabla u_{\delta,r_0} \in L^2(0, T; L^1), \end{aligned} \quad (6.6)$$

together with (6.3), yields

$$\rho_{\delta,r_0} u_{\delta,r_0} \in L^2(0, T; W^{1,1}).$$

In order to apply the Aubin-Lions Lemma, we also need to show

$$\partial_t(\rho_{\delta,r_0} u_{\delta,r_0}) \text{ is bounded in } L^2(0, T; H^{-s}), \text{ for some constant } s > 0,$$

actually, use the momentum equation (5.7)₂, it's easy to check that

$$\partial_t(\rho_{\delta,r_0} u_{\delta,r_0}) \text{ is bounded in } L^2(0, T; H^{-3}).$$

Hence, using the Aubin-Lions Lemma, the Lemma 6.3 is proved. \square

6.4. Step 4: Convergence of $\sqrt{\rho_{\delta,r_0}} u_{\delta,r_0}$.

Lemma 6.4. *We have*

$$\sqrt{\rho_{\delta,r_0}} u_{\delta,r_0} \rightarrow m/\sqrt{\rho}, \quad \text{strongly in } L^2(0, T; L^2).$$

In particular, we have $m(x, t) = 0$ a.e. on $\{x \mid \rho(x, t) = 0\}$ and there exists a function $u(x, t)$ such that $m(x, t) = \rho(x, t)u(x, t)$ and

$$\sqrt{\rho_{\delta,r_0}} u_{\delta,r_0} \rightarrow \sqrt{\rho} u, \quad \text{strongly in } L^2(0, T; L^2).$$

Proof. Recall the Lemma 6.3, we define velocity $u(x, t)$ by setting $u(x, t) = m(x, t)/\rho(x, t)$ when $\rho(x, t) \neq 0$ and $u(x, t) = 0$ when $\rho(x, t) = 0$, we have

$$m(x, t) = \rho(x, t)u(x, t).$$

Moreover, Fatou's lemma yields

$$\begin{aligned} \int_0^T \int \rho u^3 dx dt &\leq \int_0^T \int \liminf_{\delta, r_0 \rightarrow 0} \rho_{\delta, r_0} u_{\delta, r_0}^3 dx dt \\ &\leq \liminf_{\delta, r_0 \rightarrow 0} \int_0^T \int \rho_{\delta, r_0} u_{\delta, r_0}^3 dx dt, \end{aligned}$$

hence, $\rho^{\frac{1}{3}} u \in L^3(0, T; L^3)$.

Since $m_{\delta, r_0} \rightarrow m$ a.e. and $\rho_{\delta, r_0} \rightarrow \rho$ a.e., it's easy to show that

$$\sqrt{\rho_{\delta, r_0}} u_{\delta, r_0} \rightarrow m_{\delta, r_0} / \sqrt{\rho_{\delta, r_0}}, \quad \text{a.e. in } \{\rho(x, t) \neq 0\},$$

and for almost every (x, t) in $\{\rho(x, t) = 0\}$, we have

$$\sqrt{\rho_{\delta, r_0}} u_{\delta, r_0} \mathbf{1}_{|u_{\delta, r_0}| \leq M} \leq M \sqrt{\rho_{\delta, r_0}} \rightarrow 0,$$

as a matter of fact, $\sqrt{\rho_{\delta, r_0}} u_{\delta, r_0} \mathbf{1}_{|u_{\delta, r_0}| \leq M}$ converges to $\sqrt{\rho} u \mathbf{1}_{|u| \leq M}$ almost everywhere for (x, t) . Meanwhile, $\sqrt{\rho_{\delta, r_0}} u_{\delta, r_0} \mathbf{1}_{|u_{\delta, r_0}| \leq M}$ is bounded in $L^\infty(0, T; L^6)$, using the Egoroffs theorem gives

$$\sqrt{\rho_{\delta, r_0}} u_{\delta, r_0} \mathbf{1}_{|u_{\delta, r_0}| \leq M} \rightarrow \sqrt{\rho} u \mathbf{1}_{|u| \leq M} \quad \text{strongly in } L^2(0, T; L^2). \quad (6.7)$$

Since

$$\begin{aligned}
& \int_0^T \int |\sqrt{\rho_{\delta,r_0}} u_{\delta,r_0} - \sqrt{\rho} u|^2 dx dt \\
& \leq \int_0^T \int |\sqrt{\rho_{\delta,r_0}} u_{\delta,r_0} \mathbf{1}_{|u_{\delta,r_0}| \leq M} - \sqrt{\rho} u \mathbf{1}_{|u| \leq M}|^2 dx dt \\
& + 2 \int_0^T \int |\sqrt{\rho_{\delta,r_0}} u_{\delta,r_0} \mathbf{1}_{|u_{\delta,r_0}| \geq M}|^2 dx dt + 2 \int_0^T \int |\sqrt{\rho} u \mathbf{1}_{|u| \geq M}|^2 dx dt \quad (6.8) \\
& \leq \int_0^T \int |\sqrt{\rho_{\delta,r_0}} u_{\delta,r_0} \mathbf{1}_{|u_{\delta,r_0}| \leq M} - \sqrt{\rho} u \mathbf{1}_{|u| \leq M}|^2 dx dt \\
& + \frac{2}{M} \int_0^T \int \rho_{\delta,r_0} u_{\delta,r_0}^3 dx dt + \frac{2}{M} \int_0^T \int \rho u^3 dx dt \rightarrow 0,
\end{aligned}$$

as $r_0 = \delta \rightarrow 0$ and $M \rightarrow +\infty$. Thus we proved that

$$\sqrt{\rho_{\delta,r_0}} u_{\delta,r_0} \rightarrow \sqrt{\rho} u \text{ strongly in } L^2(0, T; L^2).$$

□

6.5. Step 5: Convergence of the terms $r_0 u_{\delta,r_0}$, $\rho_{\delta,r_0} \mathbb{D} u_{\delta,r_0}$, and $\rho_{\delta,r_0} \nabla \Delta^3 \rho_{\delta,r_0}$. Let $r_0 = \delta$, since $\sqrt{r_0} u_{\delta,r_0} \in L^2(0, T; L^2)$, for any test function $\varphi \in C_{per}^\infty((0, T); \mathbb{T}^3)$, we have

$$r_0 \int_0^T \int u_{\delta,r_0} \varphi dx dt \leq \sqrt{r_0} \|\sqrt{r_0} u_{\delta,r_0}\|_{L^2(L^2)} \|\varphi\|_{L^2(L^2)} \rightarrow 0, \text{ as } r_0 \rightarrow 0.$$

To deal with the diffusion term $\rho_{\delta,r_0} \mathbb{D} u_{\delta,r_0}$, recall (3.32), we have

$$\begin{aligned}
& \int_0^T \int \operatorname{div}(\rho_{\delta,r_0} \mathbb{D} u_{\delta,r_0}) \varphi = \int_0^T \int \partial_i (\rho_{\delta,r_0} (\frac{\partial_i u_{\delta,r_0}^j + \partial_j u_{\delta,r_0}^i}{2})) \varphi \\
& = \frac{1}{2} \int_0^T \int (\rho_{\delta,r_0} u_{\delta,r_0}^j) \partial_{ii} \varphi + \frac{1}{2} \int_0^T \int \partial_i \rho_{\delta,r_0} u_{\delta,r_0}^j \partial_i \varphi + \frac{1}{2} \int_0^T \int (\rho_{\delta,r_0} u_{\delta,r_0}^i) \partial_{ij} \varphi \\
& + \frac{1}{2} \int_0^T \int \partial_j \rho_{\delta,r_0} u_{\delta,r_0}^i \partial_i \varphi, \quad (6.9) \\
& = \frac{1}{2} \int_0^T \int (\sqrt{\rho_{\delta,r_0}} \sqrt{\rho_{\delta,r_0}} u_{\delta,r_0}^j) \partial_{ii} \varphi + \int_0^T \int \partial_i \sqrt{\rho_{\delta,r_0}} \sqrt{\rho_{\delta,r_0}} u_{\delta,r_0}^j \partial_i \varphi \\
& + \frac{1}{2} \int_0^T \int (\sqrt{\rho_{\delta,r_0}} \sqrt{\rho_{\delta,r_0}} u_{\delta,r_0}^i) \partial_{ij} \varphi + \int_0^T \int \partial_j \sqrt{\rho_{\delta,r_0}} \sqrt{\rho_{\delta,r_0}} u_{\delta,r_0}^i \partial_i \varphi,
\end{aligned}$$

by using Lemma 6.1 and Lemma 6.4, we can show

$$\int_0^T \int \operatorname{div}(\rho_{\delta,r_0} \mathbb{D} u_{\delta,r_0}) \varphi \rightarrow \int_0^T \int \operatorname{div}(\rho \mathbb{D} u) \varphi dx dt.$$

Finally, we show the convergence of the high order term $\rho_{\delta,r_0} \nabla \Delta^3 \rho_{\delta,r_0}$:

Since

$$\delta^{\frac{5}{14}} \|\rho_{\delta,r_0}\|_{L^{\frac{14}{5}}(H^3)} \leq \|\rho_{\delta,r_0}\|_{L^\infty(L^3)}^{\frac{2}{7}} \|\sqrt{\delta} \rho_{\delta,r_0}\|_{L^2(H^4)}^{\frac{5}{7}} \leq C < +\infty,$$

which implies $\delta^{\frac{5}{14}} \rho_{\delta, r_0} \in L^{\frac{14}{5}}(0, T; H^3)$.

For any test function $\varphi \in C_{per}^\infty([0, T]; \mathbb{T}^3)$, we have

$$\delta \int_0^T \int \rho_{\delta, r_0} \nabla \Delta^3 \rho_{\delta, r_0} \varphi dx dt = -\delta \int_0^T \int \Delta \operatorname{div}(\rho_{\delta, r_0} \varphi) \Delta^2 \rho_{\delta, r_0} dx dt,$$

we focus on the most difficult term

$$\begin{aligned} & |\delta \int_0^T \int \Delta(\nabla \rho_{\delta, r_0}) \Delta^2 \rho_{\delta, r_0} \varphi dx dt| \\ & \leq C \delta^{\frac{1}{7}} \|\sqrt{\delta} \Delta^2 \rho_{\delta, r_0}\|_{L^2(L^2)} \|\delta^{\frac{5}{14}} \nabla^3 \rho_{\delta, r_0}\|_{L^{\frac{14}{5}}(L^2)} \|\varphi\|_{L^7(L^\infty)} \rightarrow 0, \end{aligned} \quad (6.10)$$

as $\delta \rightarrow 0$.

Similarly, we can deal with the other terms from

$$\delta \int_0^T \int \Delta \operatorname{div}(\rho_{\delta, r_0} \varphi) \Delta^2 \rho_{\delta, r_0} dx dt.$$

Thus, we have

$$\delta \int_0^T \int \rho_{\delta, r_0} \nabla \Delta^3 \rho_{\delta, r_0} \varphi dx dt \rightarrow 0,$$

as $\delta \rightarrow 0$.

With all above compactness results, we can pass to the limits in (5.7) as $\delta \rightarrow 0$, we have

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= 0, \text{ holds in the sense of distribution on } (0, T) \times \mathbb{T}^3, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - \operatorname{div}(\rho \mathbb{D}u) + r_1 \rho |u| u &= \rho \nabla \Phi, \\ &\text{holds in the sense of distribution on } (0, T) \times \mathbb{T}^3, \\ \Delta \Phi &= -4\pi G(\rho - 1), \text{ holds a.e. on } (0, T) \times \mathbb{T}^3. \end{aligned} \quad (6.11)$$

Furthermore, due to the lower semi-continuity of the convex functions, we can obtain the following energy inequality and B-D entropy by passing to the limits as $r_0 = \delta \rightarrow 0$:

$$\int \frac{1}{2} \rho u^2 + \frac{1}{a\gamma(\gamma-1)} \rho^\gamma dx + \int_0^T \int \rho |\mathbb{D}u|^2 + r_1 \int_0^T \int \rho |u|^3 \leq C(T), \quad (6.12)$$

and

$$\frac{1}{2} \int \rho \left(u + \frac{\nabla \rho}{\rho}\right)^2 dx + \frac{1}{2} \int_0^T \int \rho |\nabla u|^2 + \frac{2}{a\gamma^2} \int_0^T \int |\nabla \rho^{\frac{\gamma}{2}}|^2 \leq C(T). \quad (6.13)$$

Thus we have completed the proof of the Theorem 1.1.

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